Math 5440
Mond. 10/25

We'll go over the Friday notes today.

HW for Friday 10/29

1. As usual, let $V = \{f : \mathbb{R} \to \mathbb{R},\ f \text{ is } 2\pi\text{-periodic, bounded, piecewise continuous}\}$
If f is in V, piecewise C^1 (but with jump discontinuities)
Let f have a jump discontinuity at x_0,
with left and right limits

$$f^-(x_0) = \lim_{x \to x_0^-} f(x)$$

$$f^+(x_0) = \lim_{x \to x_0^+} f(x)$$

Modify the proof on page 4 of Wed 10/20 notes to show

$$\lim_{N \to \infty} S_N(f; x_0) = \tfrac{1}{2}\left(f^-(x_0) + f^+(x_0)\right)$$

Hint: since $D_N(t) = \dfrac{\sin(N+\frac{1}{2})t}{\sin \frac{1}{2}t}$ is even,

$$\frac{1}{2\pi}\int_{-\pi}^{0} D_N(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} D_N(t)\,dt = \frac{1}{2}.$$

Use this fact to show

$$\tfrac{1}{2}\left(f^-(x_0) + f^+(x_0)\right) = \frac{1}{2\pi}\int_0^{\pi}\left(f^-(x_0) - f(x_0 - t)\right) D_N(t)\,dt + \frac{1}{2\pi}\int_{-\pi}^{0}\left(f^+(x_0) - f(x_0 - t)\right) D_N(t)\,dt$$

and then proceed analogously.

2. It is not true in general that $S_N(f;x) \to f(x)$ uniformly for 2π-periodic (only) continuous functions f. It's not even true for pointwise convergence.
Nevertheless, it is true that any such f can be approximated uniformly closely by trigonometric polynomials (just not by f's truncated Fourier series).
There is an abstract proof of this (Stone-Weierstrasse Theorem, special case), as well as a direct argument using the Fejer kernel. This is problem 15, p. 199 Rudin (last page of Wed 10/24 notes). You will show that

$$\sigma_N(f;x) = \frac{1}{N+1}\left(S_0(f;x) + S_1(f;x) + \dots + S_N(f;x)\right) \to f(x) \text{ uniformly.}$$

Hint (see HW): $$f(x) - \sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(x) - f(x-t)\right) K_N(t)\,dt = \frac{1}{2\pi}\int_{|t|<\delta} \text{"} + \frac{1}{2\pi}\int_{|t|>\delta} \text{"}$$
$$= I_1 + I_2$$

Hint cont'd:

Let $\varepsilon > 0$.

Pick δ s.t. $|I_1| < \varepsilon/2 \ \forall N$ (use f is continuous).

Then pick M s.t. $N \geq M \Rightarrow |I_2| < \varepsilon/2$.

Recall (? if you don't, we'll go over this in class)

Let $\{f_j\}$ continuous on $[a,b] \subset \mathbb{R}$, and continuously differentiable as well, i.e. $f_j \in C^1([a,b])$.

$$\sum_{j=1}^{N} f_j \to f \text{ uniformly}$$

$$\sum_{j=1}^{N} f_j' \to h \text{ uniformly}$$

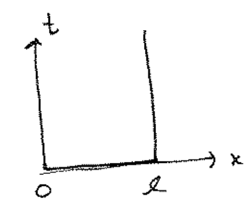
Then $f \in C^1([a,b])$ and $f' = \sum_{j=1}^{\infty} f_j'$

3) Use the analysis fact to show that the separated variable sol'ns to the Dirichlet and Neumann IVP's for the homogeneous heat equation (see page 4 of Oct. 6 notes) are infinitely differentiable in x, even if the initial temperature was only piecewise continuous, for $t > 0$

(You could use the multivariable version of the analysis theorem above to show the sol'n $u(x,t)$ is ∞'ly diff'ble (all partials exist & are cont.), for $t > 0$)

4) Also on Oct. 6 we worked out how to solve the wave eqtn IBVP's, using the building blocks below.

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$



DP: $u(0,t) = u(\ell,t) = 0$

sine series for f, g

sol's

$\left(\sin \frac{n\pi}{\ell} x\right)\left(\cos \frac{n\pi}{\ell} ct\right)$ for f

$\left(\sin \frac{n\pi}{\ell} x\right)\left(\sin \frac{n\pi}{\ell} ct\right)$ for g.

NP: $u_x(0,t) = u_x(\ell,t)$

$\left(\cos \frac{n\pi}{\ell} x\right)\left(\cos \frac{n\pi}{\ell} ct\right)$ for f

$\left(\cos \frac{n\pi}{\ell} x\right)\left(\sin \frac{n\pi}{\ell} ct\right)$ for g

for (4), explain why these Fourier series separated sol'ns recover the d'Alembert sol'ns

$$u(x,t) = \frac{1}{2}\left(\bar{f}(x+ct) + \bar{f}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}(\bar{x})\,d\bar{x}$$

for the appropriate even, odd extensions of f

5) Book problems: 4.23.1, 4.23.2