Math 5440
Monday 18 Oct.

Chapter 4 cont'd: Fourier series.

Finish discussing Friday notes, pages 3-4

Remarks • $(V, \langle\cdot,\cdot\rangle)$ with $V = \{ f: \mathbb{R} \to \mathbb{R}$ s.t. f is piecewise cont, bounded, 2ℓ-periodic$\}$

$$\langle f, g\rangle := \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)g(x)\,dx$$

Thm 4 p.4 Friday

fails the positivity requirement $\langle f,f\rangle = 0$ iff $f=0$ unless we consider the elements of V to be equivalence classes ~~of~~ of functions which agree except for at a finite number of points. $\sim$ such considerations eventually lead to Lebesgue integration & measure e.g. Math 5210

Remark Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space,

$\{u_1, u_2, \dots\}$ orthonormal

$V_n = \operatorname{span}\{u_1, u_2, \dots u_n\}$.

Then since $x \in V \Rightarrow$

$$x = \operatorname{proj}_{V_n} x + z_n \qquad z_n \perp V_n \qquad \operatorname{proj}_{V_n} x = \sum_{j=1}^{n} \langle x, u_j\rangle u_j$$

Pyth $\Rightarrow$

$$\|x\|^2 = \|\operatorname{proj}_{V_n} x\|^2 + \|z_n\|^2 \qquad \|\operatorname{proj}_{V_n} x\|^2 = \sum_{j=1}^{n} \langle x, u_j\rangle^2$$

check!

Deduce

$$\sum_{j=1}^{n} \langle x, u_j\rangle^2 \le \|x\|^2$$

$n \to \infty$:

$$\boxed{\sum_{j=1}^{\infty} \langle x, u_j\rangle^2 \le \|x\|^2}$$ Bessel's inequality

and

$$\boxed{\sum_{j=1}^{\infty} \langle x, u_j\rangle^2 = \|x\|^2 \text{ iff } \operatorname{proj}_{V_n} x \to x \text{ in } \|\cdot\|.}$$

Parseval's equality

$$\left(\text{iff } \|z_n\|^2 \to 0 \right.$$
$$\|z_n\|^2 = \|x\|^2 - \sum_{j=1}^{n}\langle x, u_j\rangle^2$$

Def: $\{u_1, u_2, \dots\}$ is called a complete orthonormal system for V if Parseval's equality holds $\forall x \in V$, i.e. $\operatorname{proj}_{V_n} x \to x \;\; \forall x$.

We will prove that $\left\{ \frac{1}{\sqrt{2}}, \cos\frac{\pi}{\ell}x, \sin\frac{\pi}{\ell}x, \dots \cos\frac{n\pi}{\ell}x, \sin\frac{n\pi}{\ell}x, \dots \right\}$

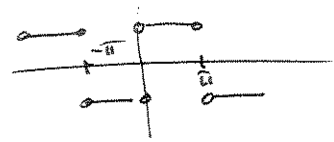
is a complete orthonormal system for over V of 2-ℓ-periodic fns (Wed?)

In this case Parseval's equality reads

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)^2\,dx$$

<u>Example</u> the 2π-periodic function

$$f(x) = \begin{cases} 1 & 0<x<\pi \\ -1 & -\pi<x<0 \end{cases}$$



is odd, so has a Fourier sine series

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin nx = \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)\Big]_0^{\pi}$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd.} \end{cases}$$

So $\quad f \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin nx$

<u>Parseval</u> $\Rightarrow \quad \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} 1\,dx}_{2\pi} = \frac{16}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}$

$$\boxed{\Rightarrow \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}}$$

magic.

but $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n \text{ odd}}\frac{1}{n^2} + \underbrace{\underbrace{\sum_{n \text{ even}}\frac{1}{n^2}}_{\sum_{k=1}^{\infty}\frac{1}{(2k)^2}}}_{\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}}$

$$\Rightarrow \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n \text{ odd}}\frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

but $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} + \sum_{n=1}^{\infty}\frac{1}{n^2}$

$$= 2\sum_{n \text{ odd}}\frac{1}{n^2}$$

$$= \frac{\pi^2}{4}$$

$$\boxed{\text{So } \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{6} = \frac{\pi^2}{12}}$$

Scaling We usually rescale the domain to reduce to 2π-periodic functions

Consider $V^{2\ell} = 2\ell$-periodic, $\langle f,g\rangle = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)g(x)\,dx$.

$T: V^{2\pi} \to V^{2\ell}$

$(Tf)(x) = f\left(\frac{\pi}{\ell}x\right)$

this is an isometry of inner product spaces:

$$\langle Tf, Tg\rangle = \frac{1}{\ell}\int_{-\ell}^{\ell} f\left(\tfrac{\pi}{\ell}x\right) g\left(\tfrac{\pi}{\ell}x\right) dx$$

subs $y = \frac{\pi}{\ell}x$; $x = \frac{\ell}{\pi}y$

$$= \frac{1}{\cancel{\ell}}\int_{-\pi}^{\pi} f(y)g(y)\frac{\cancel{\ell}}{\pi}dy = \langle f,g\rangle \quad \blacksquare$$

$$\left\{\tfrac{1}{\sqrt{2}}, \cos x, \sin x, \cos 2x, \sin 2x, \dots\right\} \xrightarrow{T} \left\{\tfrac{1}{\sqrt{2}}, \cos\tfrac{\pi}{\ell}x, \sin\tfrac{\pi}{\ell}x, \dots\right\}.$$

Complex inner product space, for Fourier series

$$\{1, e^{ix}, e^{-ix}, e^{2ix}, e^{-2ix}, \dots e^{inx}, e^{-inx}, \dots\} \qquad \langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\bar{g}(x)\,dx$$

are orthogonal, with norms $= \sqrt{2}$

...

leads to $\mathrm{proj}_{V_n} f = \sum_{k=-n}^{n} c_k e^{ikx}$ $\qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$

$c_0 = \frac{a_0}{2}$

$c_k = \frac{1}{2}(a_k - i b_k)$

relates to trigonometric Fourier series

Exercises for Friday 10/22 (in addition to those assigned Wed before break)

Recall,

<u>Def</u> An inner product space is a vector space V together with an <u>inner product</u> $\langle \cdot,\cdot \rangle : V\times V \to \mathbb{R}$ if V is real-scalar vector space, $\to \mathbb{C}$ if V is complex-scalar vector space

s.t. $\forall f,g,h \in V$, $c \in$ scalar field

$$\left.\begin{array}{ll} \text{a)} & \langle f+g,h\rangle = \langle f,h\rangle + \langle g,h\rangle \\ \text{b)} & \langle cf,h\rangle = c\langle f,h\rangle \\ \text{c)} & \overline{\langle g,f\rangle} = \langle f,g\rangle \end{array}\right\} \text{symmetric \& conjugate bilinear.}$$

in case scalars $= \mathbb{C}$.

d) $\langle f,f\rangle \geq 0$; $\langle f,f\rangle = 0$ iff $f=0$.

<u>Def</u> Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space.

a) $f \perp g$ iff $\mathrm{Re}\langle f,g\rangle = 0$ perpendicular vectors

b) $\|f\| := \langle f,f\rangle^{1/2}$ norm.

1) Let $V = \mathbb{C}^n$. Define $\langle \vec{z},\vec{w}\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$

a) Show $(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$ is an inner product space

b) Let $\vec{x},\vec{y},\vec{u},\vec{v} \in \mathbb{R}^n$. Consider also the vectors $(\vec{x},\vec{y}) \in \mathbb{R}^{2n}$, $(\vec{u},\vec{v}) \in \mathbb{R}^{2n}$

Show $\vec{x}+i\vec{y} \perp \vec{u}+i\vec{v}$ in $\mathbb{C}^n$ iff $(\vec{x},\vec{y}) \perp (\vec{u},\vec{v})$ in $\mathbb{R}^{2n}$

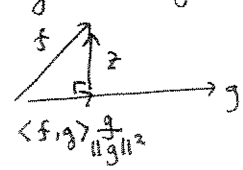
2) Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. Show

(i) $\|cf\| = |c|\,\|f\|$ c scalar

(ii) $\|f+g\|^2 = \|f\|^2 + \|g\|^2$ iff $f \perp g$ (Pythag. thm.)

(iii) $|\langle f,g\rangle| \leq \|f\|\,\|g\|$; equality iff f,g are scalar multiples (Cauchy-Schwartz)

hint: consider



(iv) $\|f+g\| \leq \|f\| + \|g\|$ (triangle inequality) hint: square both sides and apply Cauchy-Schwartz

(v) Define the distance function $d: V\times V \to \mathbb{R}$ by $d(f,g) = \|f-g\|$. Then $\forall f,g,h \in V$

a) $d(f,h) \leq d(f,g) + d(g,h)$ triangle inequality

b) $d(f,g) \geq 0$. $d(f,g) = 0$ iff $f=g$

3) a) Show that the 2π-periodic extension of $f(t) = t \quad -\pi < t < \pi$ has Fourier sine series

$$t \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt$$

b) Deduce by integration that the 2π-periodic extension of $f(t) = t^2/2$ has convergent Fourier series

$$\frac{t^2}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt + C_1 \qquad -\pi < t < \pi$$

and deduce C_1 by integrating both sides from $-\pi$ to π (or 0 to π).

c) evaluate (b) @ $t = 0$ to deduce

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

d) Integrate (b) twice more to get

$$\frac{1}{24} t^4 = \frac{\pi^2 t^2}{12} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \cos nt + C_2$$

deduce C_2 as in (b)

e) Prove

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad ; \quad \sum_{\substack{n=1,3,\ldots \\ (n \text{ odd})}} \frac{1}{n^4} = \frac{\pi^4}{96} \quad ; \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$$