Math 5440
Monday 11/29

The <u>spectral theorem</u> approach to orthonormal, complete, eigenfunction of Laplacian "bases" for $L^2(\Omega)$

Recall/review: (we'll refer to these as needed.)

<u>Def</u> H is a (real scalar) Hilbert space if it is a (real scalar) <u>inner product space</u> and it is <u>complete</u> in the metric space sense that Cauchy sequences coverge to a limit in the space.

<u>examples</u>
- $\mathbb{R}^n$, $\langle\,,\,\rangle$ = dot product. You've proven $\mathbb{R}^n$ is complete in 3220.
- $\ell^2 = \left\{ \text{sequences } \vec{x} = \{x_k\}_{k=1}^{\infty}, \text{ with } \sum_{k=1}^{\infty} x_k^2 < \infty. \right\}$
  $$\langle \vec{x}, \vec{y} \rangle = \sum_{k=1}^{\infty} x_k y_k$$
  it's a good exercise to prove ℓ^2 is complete.
- $L^2(\Omega) = \left\{ f:\Omega\to\mathbb{R} \text{ s.t. } \int_\Omega f^2 dV_n < \infty \right\}, \quad \langle f,g\rangle = \int_\Omega f(x)g(x)\,dV_n.$
  here $\Omega \subset \mathbb{R}^n$.
  you can think of $L^2(\Omega)$ as the metric space completion of functions which are square integrable in the Riemann integral; it's also the completion of continuous functions which are square integrable. The space is defined more concretely using Lebesgue integration.
- $\left\{ f:[\alpha,\beta]\to\mathbb{R} \text{ s.t. } \int_\alpha^\beta \rho f^2 dx < \infty \quad \text{where } \rho>0 \text{ on } (\alpha,\beta). \right\}$
  $\langle f,g\rangle = \int_\alpha^\beta \rho f g\,dx.$ Call this space $L^2([\alpha,\beta],\rho)$.

<u>Def</u> Let H be a Hilbert space. A collection $\{u_k\}_{k\in\mathbb{N}}$ of orthonormal vectors is called a (Hilbert) basis for H if the metric space closure of

$$\text{span}\{u_k\} = \{\text{finite linear combos of } u_k\}$$

is all of H.

This is equivalent to

(1) $$\text{proj}_{V_N} x := \sum_{k=1}^{N} \langle x, u_k\rangle u_k \xrightarrow{N\to\infty} x \qquad \forall x\in H$$

and to

(2) $$\sum_{k=1}^{\infty} \langle x, u_k\rangle^2 = \|x\|^2 \qquad \forall x \in H \qquad \text{(Parseval's equality).}$$

and to

(3) $$L: H \to \ell^2, \quad x \longmapsto \{\langle x,u_k\rangle\} \qquad \text{is a linear isometry.}$$

<u>Def</u>: $T: H \to H$ linear is <u>self-adjoint</u> iff $\langle Tf, g\rangle = \langle f, Tg\rangle \;\forall f, g \in H$.

<u>examples</u>

1) $T: \mathbb{R}^n \to \mathbb{R}^n \quad T(\vec{x}) = A\vec{x}$ where $A^T = A$

Since $(A\vec{x})\cdot\vec{y} = (Ax)^T y = x^T A^T y = x^T A y = \vec{x}\cdot A\vec{y}$.

$\left(= \sum_{i,j} x^i y^j a_{ij} \right)$

2) $T: L^2(\Omega) \to L^2(\Omega)$

$(Tf)(x) = \int_\Omega G(x,\xi) f(\xi)\, dV_n(\xi)$. Where $G(x,\xi)$ is the Green's fcn for Δ & Dirichlet problem in Ω.

$\langle Tf, g\rangle = \int_\Omega \left(\int_\Omega G(x,\xi) f(\xi)\, d\xi \right) g(x)\, d\cancel{x}$

$= \iint_{\Omega\Omega} G(x,\xi) f(\xi) g(x)\, d\xi\, dx = \langle f, Tg\rangle$ because $G(x,\xi) = G(\xi, x)$

3) $T: L^2([\alpha,\beta], \rho) \to L^2([\alpha,\beta],\rho)$

$Tf(x) = \frac{1}{\rho(x)} \int_\alpha^\beta G(x,\xi) f(\xi)\, d\xi$

with G the Green's function for $(pu')' + qu = L(u)$.

$\langle Tf, g\rangle = \int_\alpha^\beta \frac{1}{\rho(x)} \int_\alpha^\beta G(x,\xi) f(\xi)\, d\xi\, g(x) \rho(x)\, dx$

$= \int_\alpha^\beta \int_\alpha^\beta G(x,\xi) f(\xi) g(x)\, dx\, d\xi = \langle f, Tg\rangle$

<u>Def</u> $T: H \to H$ linear is <u>compact</u> iff $\forall \{f_k\} \subset H$ bounded, $\{Tf_k\}$ has a convergent subsequence

<u>examples</u> 1) above: all matrix transformations of $\mathbb{R}^n$ are compact.

3) we'll show (later) that the 1-d Green's fcn transformation is compact

2) the higher dim'l Green's fcn transformation for Laplace operator

$f \longmapsto u$ where $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

is compact. Proof involves <u>Sobolev space theory</u>, beyond scope of this class

<u>Def</u>. Let $T: H \to H$ be linear.

The <u>operator norm</u> $\|T\|_{op} := \sup_{x \neq 0} \frac{|Tx|}{|x|} = \sup_{x \neq 0} |T(\frac{x}{|x|})| = \sup_{|x|=1} |Tx|$

<u>Theorem</u> T is continuous iff $\|T\|_{op} < \infty$.

<u>proof</u>: $\Rightarrow$: If T is continuous then it is continuous at $x_0 = 0$.

For $\varepsilon = 1$ pick δ s.t.

$$|x| \leq \delta \Rightarrow |Tx| < \varepsilon$$

Thus for $|x| \leq 1$, $|T(x)| = \frac{1}{\delta} |T(\delta x)| < \frac{\varepsilon}{\delta}$

$$\Rightarrow \|T\|_{op} \leq \varepsilon / \delta$$

$\Leftarrow$: notice $|T(x)| \leq \|T\|_{op} |x|$ holds forall x.

thus $|T(x) - T(y)| = |T(x-y)| \leq \|T\|_{op} |y-x|$.

Thus T is Lipschutz continuous with Lipschutz constant $L = \|T\|_{op}$.

<u>Theorem</u> T compact $\Rightarrow$ T continuous.

<u>pf</u>: ~~If T is compact and~~ Let $\|T\|_{op} = \infty$

thus $\exists \{x_k\}$, $|x_k| = 1$, $|T(x_k)| \to \infty$.

thus sequence $\{T(x_k)\}$ has no convergent subseq.

$\Rightarrow$ T is not compact

so not cont $\Rightarrow$ not compact

$\blacksquare$

Spectral theorem for compact self adjoint operators on a Hilbert space
(see wikipedia or e.g. "Real Analysis" by Serge Lang for more general versions)

<u>Theorem</u> Let $T : H \to H$ be a compact self-adjoint linear transformation on the real Hilbert Space H. Then

(i) eigenspaces of T with different eigenvalues are mutually perpendicular

like 2270 $\longrightarrow$ (ii) if $\dim H < \infty$ $\exists$ basis of H consisting of orthonormal eigenvectors

(iii) if $\dim H = \infty$ and $\ker T$ (which is the $\lambda = 0$ eigenspace) has finite dimension then $\exists$ a complete orthonormal Hilbert basis of eigenvectors of T, $\{f_k\}_{k \in \mathbb{N}}$, $Lf_k = \lambda_k f_k$, $\lambda_k \to 0$ as $k \to \infty$.

proof:

(i) If $Tf = \lambda_1 f$, $Tg = \lambda_2 g$

then $\langle Tf, g\rangle = \langle f, Tg\rangle \Rightarrow \langle \lambda_1 f, g\rangle = \langle f, \lambda_2 g\rangle = \lambda_1 \langle f, g\rangle = \lambda_2 \langle f, g\rangle$

so either $f \perp g$ or $\lambda_1 = \lambda_2$ ∎

(ii) (<u>not</u> the 2270 proof).

use induction. If $\dim H = 1$, $H = \{tf\}_{t \in \mathbb{R}}$ where f is a unit vector basis.

$Tf = \lambda f$ for some $\lambda \in \mathbb{R}$ ∎

(so $\{f\}$ is an orthonormal eigenbasis for H)

now assume theorem is true for all Hilbert Spaces of dim. $n-1$, and let H have dimension n.

consider
$$\sup_{g \neq 0} \frac{|\langle Tg, g\rangle|}{\langle g, g\rangle} = \sup_{g \neq 0} \left|\left\langle T\left(\tfrac{g}{\|g\|}\right), \tfrac{g}{\|g\|}\right\rangle\right| = \sup_{|h|=1} |\langle Th, h\rangle|.$$

because $F(h) := |\langle Th, h\rangle|$ is continuous function and because the unit sphere in H is compact $\left(= \left\{ \sum_{i=1}^{n} c_i u_i \text{ s.t. } \sum c_i^2 = 1 \right\} \right.$ if $\{u_1, \dots u_n\}$ is an orthonormal basis of H.)

this suprememum is a max, so pick f_1 a unit vector with

$$\langle Tf_1, f_1\rangle = \lambda_1. \qquad |\lambda_1| = \sup_{|h|=1} \langle Th, h\rangle$$

Let $W = (\text{span}\{f_1\})^{\perp}$, the $n-1$ dim'l orthogonal complement.

Let $w \in W$, $\varepsilon \in \mathbb{R}$

then $\dfrac{\langle T(f_1+\varepsilon w), f_1+\varepsilon w\rangle}{\langle f_1+\varepsilon w, f_1+\varepsilon w\rangle}$ has a local max (if $\lambda_1 > 0$) or a local min (if $\lambda_1 < 0$) at $\varepsilon = 0$. (If $\lambda_1 = 0$, T is the zero transformation and pick any orthormal basis of H to be the $\lambda = 0$ eigenbasis.)

$$\| $$

$$\frac{\langle Tf_1, f_1\rangle + 2\varepsilon\langle Tf_1, w\rangle + \varepsilon^2\langle Tw, w\rangle}{\langle f_1, f_1\rangle + \varepsilon^2\langle w, w\rangle}.$$

$$\frac{d}{d\varepsilon}(\%)\Big|_{\varepsilon=0} = 2\langle Tf_1, w\rangle = 0$$ because local max or min.

<u>deduce</u> $Tf_1 \perp w \ \forall w \in W$

$\Rightarrow Tf_1 \,/\!/\, f_1$, and since $\langle Tf_1, f_1\rangle = \lambda_1$ deduce $\boxed{Tf_1 = \lambda_1}$

but $\langle Tf_1, w\rangle = \langle f_1, Tw\rangle$

so also deduce $Tw \perp f_1 \ \forall w \in W$

i.e. $T: W \to W$.

by induction $\exists$ o.n. eigenbasis of W, $\{f_2, \dots f_n\}$
$Tf_j = \lambda_j f_j$.

Thus $\{f_1, f_2, \dots f_n\}$ is orthonormal eigenbasis of H

(iii) $\dim H = \infty$.

modify finite dim'l proof.

Let $\alpha = \sup_{g \neq 0} \dfrac{|\langle Tg, g\rangle|}{\langle g, g\rangle} = \sup_{|g|=1} |\langle Tg, g\rangle|$.

<u>Lemma</u> $\alpha = \|T\|_{op}$

proof: $\|T\|_{op} = \sup_{|g|=1} |Tg|$

$= \sup_{\substack{|g|=1 \\ |h|=1}} \langle Tg, h\rangle$ $\quad$ ∵ C.S. $|\langle Tg, h\rangle| \leq |Tg||h| = |Tg|$

but $\langle Tg, \frac{Tg}{|Tg|}\rangle = \frac{|Tg|^2}{|Tg|} = |Tg|$.

$|\langle T(g+h), g+h\rangle| = |\langle Tg, g\rangle + \langle Th, h\rangle + 2\langle Tg, h\rangle| \leq \alpha |g+h|^2$

$|\langle T(g-h), g-h\rangle| = |\langle Tg, g\rangle + \langle Th, h\rangle - 2\langle Tg, h\rangle| \leq \alpha |g-h|^2$

$|4\langle Tg, h\rangle| = |\langle T(g+h), g+h\rangle - \langle T(g-h), g-h\rangle|$

$\leq | \quad | + | \quad |$

$= \alpha(|g+h|^2 + |g-h|^2)$

$= \alpha(2|g|^2 + 2|h|^2)$

$= 4\alpha$.

$\Rightarrow \langle Tg, h\rangle \leq \alpha$

$\boxed{\Rightarrow \|T\|_{op} \leq \alpha.}$

but, for $|g| = 1$, $|\langle Tg, g\rangle| \leq |Tg||g|$ C.S.

$\leq \|T\||g|^2$

$= \|T\|_{op}$

so $\boxed{\alpha \leq \|T\|_{op}.}$ ∎

now try to proceed as in finite dim'l case.

Let $\{g_k\}$, $|g_k| = 1$, $\langle Tg_k, g_k\rangle \to \lambda_1$, $|\lambda_1| = \alpha$.

- Since $\{g_k\}$ is bounded and T is compact a subsequence of $\{Tg_k\}$ converges

renumber, assume

$\langle Tg_k, g_k\rangle \to \lambda_1$

$\{Tg_k\} \to g$.

$\lambda_1 \neq 0$, since $|\lambda_1| = \alpha$ and $\alpha = 0 \Rightarrow T \equiv 0$ which contradicts $\dim(\ker T) < \infty$.

consider

$$\| Tg_k - \lambda_1 g_k \|^2 = |Tg_k|^2 - 2\lambda_1 \langle Tg_k, g_k \rangle + \lambda_1^2$$

$$\leq \alpha^3 - 2\lambda_1 \underbrace{\langle Tg_k, g_k \rangle}_{\downarrow \ \lambda_1} + \alpha^2$$

so LHS ≤ 0 in the limit.

$$\Rightarrow |\underbrace{Tg_k}_{\downarrow \ g} - \lambda_1 g_k|^2 \to 0$$

$$\Rightarrow g - \lambda_1 g_k \to 0$$

$$\Rightarrow g_k \to \frac{1}{\lambda_1} g.$$

$$T\left(\frac{1}{\lambda_1} g\right) = T\left(\lim_{k\to\infty} g_k\right) = \lim_{k\to\infty} Tg_k = g$$

$$\boxed{Tg = \lambda_1 g} \qquad \text{Call } g = f_1$$

Let $H_1 = (\text{span}\{g\})^\perp$.

as before T maps $H_1 \to H_1$ and remains compact and self adjoint on this sub-Hilbert space.

induct; construct orthonormal eigenvectors

$$f_1, f_2, \ldots f_n, \ldots$$

$$Tf_j = \lambda_j f_j$$

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \ldots \geq |\lambda_k| \geq |\lambda_{k+1}| > 0.$$

<u>Claim 1</u>: $\lambda_k \to 0$

pf: if not $\exists$ subseq. $\{\lambda_{k_j}\}$, $|\lambda_{k_j}| \geq \varepsilon$ $\forall j$. some $\varepsilon > 0$.

but then $\{f_{k_j}\}$ violates compactness of T, since

$$|\underbrace{Tf_{k_j}}_{= \lambda_{k_j} f_j} - \underbrace{Tf_{k_\ell}}_{= \lambda_{k_\ell} f_\ell}|^2 = \lambda_{k_j}^2 + \lambda_{k_\ell}^2 \geq 2\varepsilon^2.$$

can have no convergent subsequence.

<u>Claim 2</u>: the collection $\{f_j\}$ union an orthormal basis for $\ker T$ is a Hilbert basis for H.

(this takes a couple more pages to explain.)

~ ran out of prep time today.

Class exercises for Fri, cont'd

III Show ℓ^2 is complete (see Def on page 1)

IV Show the 4 definitions of a Hilbert basis consisting of orthonormal vectors on page 1 are equivalent

V Show if $T\vec{x} = A\vec{x}$ is a linear transformation of $\mathbb{R}^n$, with dot product as inner product, then

(a) if A is symmetric, $\|T\|_{op} = \max_i |\lambda_i|$, where λ_i are the eigenvalues of A.

(b) in general, $\|T\|_{op}^2 = $ max eigenvalue of $A^T A$.

(you may use the finite dim'l spectral theorem to prove these claims.)

(c) What is $\|T\|_{op}$ for $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$?

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