Math 5440
Wed 11/24

Eigenfunctions of the Laplacian (and other self adjoint operators)
(connects to chapters 6-7 in Weinberger)

Overview:

Recall for wave & heat eqtns in arbitrary space dimension
- Duhamel $\Rightarrow$ solve homog PDE yields integral formulas for non-homog. PDE
- If you try separation of variables in bounded domains you end up wanting to use eigenfunctions of Δ (or $-\Delta$), with appropriate Dirichlet or Neumann BC's:

$$u_t - k\Delta u = 0$$
if $u = X(x)T(t)$
$$T'X - kT\Delta X = 0$$
$$\frac{\Delta X}{X} = \frac{T'}{kT} = -\lambda$$

$$u_{tt} - c^2\Delta u = 0$$
if $u = X(x)T(t)$
$$T''X - c^2 T\Delta X = 0$$
$$\frac{\Delta X}{X} = \frac{T''}{c^2 T} = -\lambda$$

notice $\lambda \geq 0$ so $-\lambda \leq 0$ because if $X(x)$ solves
$$\Delta X = -\lambda X$$
then
$$\int_\Omega |\nabla X|^2 + \underbrace{X\Delta X}_{-\lambda X^2} = \int_{\partial\Omega} X\frac{\partial X}{\partial n} \overset{\shortparallel}{0} \text{ for NP \& DP}$$
so
$$\lambda = \frac{\int_\Omega |\nabla X|^2}{\int_\Omega X^2}.$$

thus your separated solutions will behave as you expect (exponentially decaying or const for HE, waves for wave eqtn.)

- to solve IBVP's with DP or NP

HE $u(x,0) = f(x) \quad x \in \Omega$

WE $u(x,0) = f(x)$
$u_t(x,0) = g(x)$

need to know whether the eigenfunctions are "complete", and maybe other stronger convergence properties.

Theorem Let $\Omega \subset \mathbb{R}^n$, bounded, piecewise smooth boundary.

Then there is an orthonormal complete collection of Dirichlet eigenfunctions

$$\{\psi_k\}_{k \in \mathbb{N}},$$

$$-\Delta \psi_k = \lambda_k \psi_k$$

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \lambda_{k+1} \to \infty$$

with respect to $\langle f, g \rangle = \int_\Omega fg \, dV_n$

$\| f - \mathrm{proj}_{V_N} f \| \to 0$ as $N \to \infty$.

(Similarly for Neumann eigenfunctions) $0 = \mu_1 < \mu_2 \le \mu_3 \cdots$

We'll sketch this proof, as a special case of the spectral theorem for compact self adjoint operators on Hilbert Spaces

Hilbert Spaces: complete inner product spaces

separable: $\exists$ countable dense subset.

self adjoint operators: $L: V \to V$

$\langle u, Lv \rangle = \langle Lu, v \rangle$

$\forall u, v \in X$.

L is compact (linear) transformation, $L: V \to V$, if $\forall \{f_k\} \in V$, a bounded sequence ($\exists M > 0$ s.t. $\| f_k \| \le M \ \forall k$) the image sequence $\{L(f_k)\}$ has a convergent subsequence.

(If V is finite dim'l and L is linear it's automatically compact.)

Remark: Our $V = L^2(\Omega) = \{ f: \Omega \to \mathbb{R} \text{ s.t. } \int_\Omega f(x)^2 < \infty \}$.

Our $L \ne \Delta$, which isn't even defined on all of V.

Rather it's "Δ^{-1}", i.e. for $F \in L^2(\Omega)$ $\Delta^{-1}_{DP}(F)$ is the soln to u

$$\begin{cases} \Delta u = F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

as given either by the variational or Green's function formulations.

Examples we know of self adjoint operators.

- $\mathbb{R}^n$, $L(\vec{x}) = A\vec{x}$, A symmetric, usual dot product
  $\langle Ax, y\rangle = (Ax)^T y = x^T A^T y = x^T A y = \langle x, Ay\rangle$.

- Δ is self-adjoint on the subspace of $L^2(\Omega)$ given by
  $V_{DP} = \{ u \in L^2(\Omega)$ s.t. $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $u = 0$ on $\partial\Omega \}$
  or
  $V_{NP} = \{ u \in L^2(\Omega)$ s.t. $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega \}$

because

$$\langle u, Lv\rangle - \langle Lu, v\rangle = \int_\Omega u\Delta v - \Delta u\, v = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} = 0 \quad \text{if } u,v \in V_{DP} \text{ or if } u,v \in V_{NP}.$$

examples of complete eigenfcn collections

- $\{\sin nx\}_{n\in\mathbb{N}}$ DP eigenfcns of Δ on $[0,\pi]$
  $\langle f, g\rangle = \frac{2}{\pi}\int_0^\pi f(x)g(x)\,dx$
- $\{\cos nx\}_{n\in\mathbb{N}} \cup \{\frac{1}{\sqrt{2}}\}$ NP eigenfcns for Δ.

- $L(u) = \frac{1}{\rho(x)}\left[\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u\right]$ for fcns defined on the interval $\alpha \le x \le \beta$
  $\rho > 0$
  and $\langle u, v\rangle = \int_\alpha^\beta u(x)v(x)\rho(x)\,dx$.

(these sorts of L arise when we consider disk, ball or cylinder domains and separate variables for Δ.)

$$\langle Lu, v\rangle = \int_\alpha^\beta \frac{1}{\cancel{\rho}}[(pu')' + qu]v\cancel{\rho} = pu'v\Big]_\alpha^\beta + \int_\alpha^\beta quv - pu'v'\,dx$$

$$\langle u, Lv\rangle = \int_\alpha^\beta \frac{1}{\cancel{\rho}}[(pv')' + qv]u\cancel{\rho} = pv'u\Big]_\alpha^\beta + \int_\alpha^\beta qvu - pv'u'\,dx$$

<u>Spectral Theorem</u>: Let $(V, \langle , \rangle)$ be a Hilbert Space

$L: V \to V$ linear and compact, and self-adjoint.

Then $\exists$ a complete orthonormal basis $\{u_k\}_{k \in \mathbb{N}}$ of eigenvectors of L.

$$L u_k = \lambda_k u_k.$$

$$\lambda_k \to 0 \text{ as } k \to \infty.$$

proof on Monday.

HW for Fri Dec. 3

6.31.3 (double sine series in a square)
6.32.1, 4, 5 (soltns to homog. Laplace, heat, wave eqtns in rectangles)

(I) Also, find the fully separated soltions to heat and wave equation in the unit disk, i.e.

$$u(\vec{x}, t) = R(r)\,\Theta(\theta)\,T(t)$$

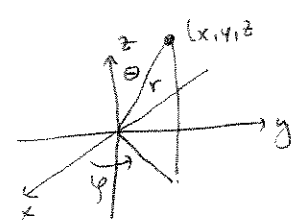
for $u_t - k\Delta u = 0$ (relates to §7.41)

$u_{tt} - c^2 \Delta u = 0$

Do not try to solve the ODE's for $R(r)$

(II) Same question in the unit ball, using spherical coords

$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$ ← not like in m.v. calc.
$z = r\cos\theta$



<u>Step 1</u>: derive $L = \Delta$ in spherical coords.