Math 5440
Fri 11/19
Chapter 7 Strauss cont'd

Problem Session Monday here (LCB 222), 4-5:30
HW is at end of today's notes and is due Wednesday

- Recall our rotationally symmetric fundamental soltns for Δ in $\mathbb{R}^n$

  write $|\vec{x}|=r$ in each space dim

  then

  $K(x) = \frac{1}{2}r \quad n=1$

  $K(x) = \frac{1}{2\pi}\ln r \quad n=2$

  $K(x) = -\frac{1}{4\pi}\frac{1}{r} \quad n=3$

  $K(x) = \alpha_n r^{2-n} \quad n\geq 3$

  $= \nabla K\cdot\hat{n}$

  α_n is chosen so that $\displaystyle\int_{S_R^{n-1}(0)} \frac{\partial K}{\partial r} = 1 \quad \forall R>0$

  i.e. $\frac{\partial K}{\partial r} = \frac{1}{|S_r^{n-1}|}$ $\forall r$.

- Recall integration by parts formula for Δ:

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}$$

We used this formula with translations of K to prove <u>mean value property</u> for harmonic functions, and a representation for a harmonic fcn in terms of its boundary data.

We can do more

<u>Def</u>: (following Strauss) The Green's function $G(x,x_0)$, $G:\bar{\Omega}\times\Omega\to\mathbb{R}$ for $-\Delta$ satisfies

(i) $G(x,x_0)\in C^2(\Omega\setminus\{x_0\})\cap C^1(\bar{\Omega}\setminus\{x_0\})$ in the x-variable, $\Delta_x G(x,x_0)=0 \quad x\neq x_0$

(ii) $G(x,x_0)=0 \quad \forall\, x\in\partial\Omega$

(iii) $G(x,x_0) = K(x-x_0) + w(x,x_0)$ where $w(x,x_0)\in C(\bar{\Omega})\cap C^2(\Omega)$ is harmonic in Ω.

<u>Remark</u> So $w(x,x_0)$ solves

$$\begin{cases} \Delta_x w(x,x_0) = 0 & \text{in } \Omega \\ w(x,x_0) = -K(x-x_0) & x\in\partial\Omega \end{cases}$$

<u>Theorem</u> Green's functions are <u>unique</u>, <u>symmetric</u>, and <u>they exist</u>

↑ easy. why?

↑ for $x_1, x_0\in\Omega$ $G(x_1,x_0)=G(x_0,x_1)$ <u>see next page</u>

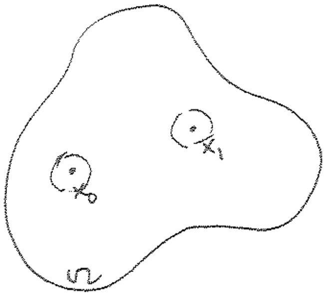
↑ true in general; findable for special geometries. (General existence proof is difficult.)

Symmetry of Green's fcn: Let $x_0, x_1 \in \Omega$. Show $G(x_0,x_1) = G(x_1,x_0)$.

Let $u(x) = G(x,x_0) = K(x-x_0) + w_1(x)$

$v(x) = G(x,x_1) = K(x-x_1) + w_2(x)$

Let $\varepsilon > 0$ s.t. $B_\varepsilon(x_0), B_\varepsilon(x_1) < \Omega$.



$$0 = \int_{\Omega \setminus (B_\varepsilon(x_0) \cup B_\varepsilon(x_1))} u\Delta v - v\Delta u = \int_{\partial\Omega} \cancelto{0}{u}\frac{\partial v}{\partial n} - \cancelto{0}{v}\frac{\partial u}{\partial n} - \int_{S_\varepsilon(x_0)} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}$$

$$- \int_{S_\varepsilon(x_1)} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}$$

$$\begin{array}{ccccccccc} 0 = & -\int_{S_\varepsilon(x_0)} u\frac{\partial v}{\partial n} & + & \int_{S_\varepsilon(x_0)} v\left(\frac{\partial K}{\partial n} + \frac{\partial w_1}{\partial n}\right) & - & \int_{S_\varepsilon(x_1)} u\left(\frac{\partial K}{\partial n} + \frac{\partial w_2}{\partial n}\right) & + & \int_{S_\varepsilon(x_1)} v\frac{\partial u}{\partial n} \\ & \Big\downarrow {\scriptstyle \varepsilon\to0} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ & 0 & & v(x_0) & & -u(x_1) & & 0 \\ & & & \shortparallel & & \shortparallel & & \\ & & & G(x_0,x_1) & & -G(x_1,x_0) & & \end{array}$$

∎

<u>**Theorem**</u> Let $G(x,x_0)$ be the Green's function for $-\Delta$, for Ω.
Then the solution to

$$\begin{cases} \Delta u = F & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$$

is

$$u(x) = \int_\Omega F(\xi)G(x,\xi)\overset{\text{n-dim'l}}{d\xi} + \int_{\partial\Omega} h(\xi)\nabla_\xi G(x,\xi)\cdot\hat{n}\overset{\text{n-1 dim'l}}{d\xi}$$

(I used Weinberger notation)

First term ↑ solves
$\Delta u_1 = F$ in Ω
$u_1 = 0$ on $\partial\Omega$
this term generalizes 1-d Green's fcn for Δ.

Second term ↑ solves
$\Delta u_2 = 0$ in Ω
$u_2 = h$ on $\partial\Omega$
this term generalizes Poisson integral formula

actually proving this theorem is quite technical.... (uses ideas like we've discussed)

but we can prove the weaker fact that if u solves $\begin{cases} \Delta u = F & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$

then the representation on previous page is valid.

We'll return to Strauss notation, prove that for $x_0 \in \Omega$

$$u(x_0) = \int_{\Omega} F(x)\, G(x,x_0)\, dV_n(x) + \int_{\partial\Omega} h(x)\, \nabla_x G(x,x_0)\, dV_{n-1}(x)$$

proof: let $v(x) = G(x,x_0)$, $\varepsilon > 0$

$$\int_{\Omega \setminus B_\varepsilon(x_0)} u\,\overset{0}{\cancel{\Delta v}} - v\,\Delta u = \int_{\Omega \setminus B_\varepsilon(x_0)} -G(x,x_0)\,F(x) \xrightarrow{\varepsilon \to 0} \boxed{\int_{\Omega} -G(x,x_0)\,F(x)}$$

$$\| $$

$$\int_{\partial\Omega} u\frac{\partial v}{\partial n} - \overset{0}{\cancel{v}}\frac{\partial u}{\partial n} \;-\; \int_{S_\varepsilon(x_0)} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}$$

$$\int_{\partial\Omega} u\frac{\partial v}{\partial n} = \boxed{\int_{\partial\Omega} h(x)\, \nabla_x G(x,x_0)}$$

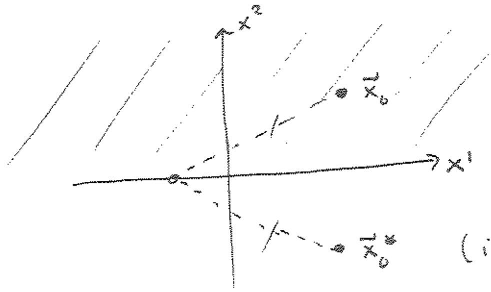
$$-\int_{S_\varepsilon(x_0)} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} = -\int_{S_\varepsilon(x_0)} u\frac{\partial v}{\partial n} + \int_{S_\varepsilon(x_0)} v\frac{\partial u}{\partial n}$$

$$\int_{S_\varepsilon(x_0)} v\frac{\partial u}{\partial n} \xrightarrow{\text{as } \varepsilon \to 0} 0$$

$$-\int_{S_\varepsilon(x_0)} u\frac{\partial v}{\partial n} = -\fint_{S_\varepsilon(x_0)} u \xrightarrow{\varepsilon \to 0} \boxed{-u(x_0)}$$

Examples of Green's functions $n=2$ (these ideas generalize to $n \geq 2$, using the corresponding $K(\vec{x})$'s.). (Strauss does $n=3$ in §7.4.)

1) upper half plane. (this is an unbounded domain, but everything can be made rigorous if you restrict to bounded solutions of Laplace equation).

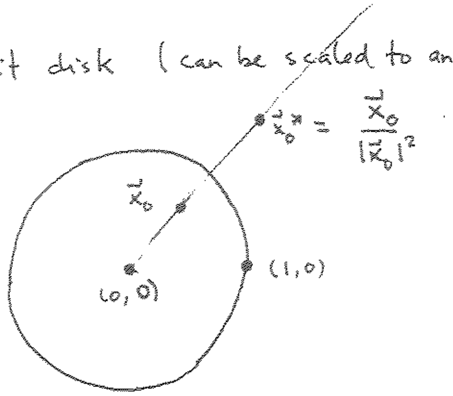


$$G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\left(\ln|\vec{x}-\vec{x}_0| - \ln|\vec{x}-\vec{x}_0^*|\right)$$

- correct singularity @ x_0
- harmonic for $\vec{x} \neq \vec{x}_0$, $\vec{x} \in \mathbb{H}$
- $=0$ for $\vec{x} \in \partial\mathbb{H}$.

(if $\vec{x}_0 = (x_0, y_0)$
$\vec{x}_0^* = (x_0, -y_0)$ is reflection across x^1-axis.

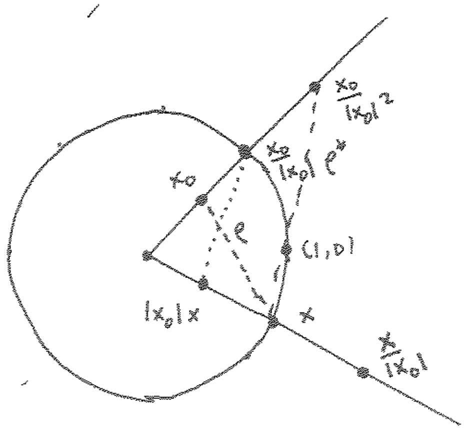
2) unit disk (can be scaled to any radius)



$$G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\left(\ln|\vec{x}-\vec{x}_0| - \ln(|x_0||\vec{x}-\vec{x}_0^*|)\right)$$

$(\vec{x}_0 = 0,\ G(\vec{x},\vec{0}) = \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0|)$.

- correct sing.
- harmonic $\vec{x} \neq \vec{x}_0$
- $=0$ on $S_1(0)$ is geometry!



for $x \in S^1$
does $|x_0|\rho^* = \rho$?

ans:

$$\rho = |x - x_0| = \left|\frac{x_0}{|x_0|} - |x_0|x\right|$$
$$= |x_0|\left|\frac{x_0}{|x|^2} - x\right| \checkmark$$

Homework for Wed. Nov. 24

From Chapter 7 Strauss:

§7.1 p174-176

#1 ← do this in $\mathbb{R}^n$, and prove the strong maximum principle for harmonic functions: If Ω is bounded, open, connected,

$$u \in C^2(\Omega) \cap C(\bar{\Omega}), \quad \Delta u = 0$$

then the max. of u occurs on $\partial\Omega$. If it occurs at any (interior) point $x_0 \in \Omega$, then u is constant.

#3, 5 (re #5, we'll discuss Dirichlet's principle on Monday; it's also discussed in the text.)

§7.4 p186-188

#6, 11, 17

Class exercises

1. We proved the mean value property (MVP) for harmonic functions with concentric spherical averages in class. Prove the MVP for balls.

i.e. if $B_R(x_0) \subset \Omega$, $\Delta u = 0$ in Ω, then

$$u(x_0) = \fint_{B_R(x_0)} u(x)\, dV_n .$$

Hint: Let $v = R^2 - x^2$, use integration by parts.

2. This one is challenging! Let $f \in C_c^2(\mathbb{R}^n)$

↑ compactly supported, i.e. $\overline{\{x \mid f(x) \neq 0\}}$ is bounded.

Define

$$u(x) = \int_{\mathbb{R}^n} f(x-\xi) K(\xi)\, d\xi = f * K(x)$$

$\mathbb{R}^n$ ← really over a bounded domain

2a). Check $u(x) = \int_{\mathbb{R}^n} f(\xi) K(x-\xi)\, d\xi$ as well ($f * K = K * f$).

2b) <u>Prove</u> $\Delta u(x) = f(x)$, using (2a). (this justifies the concept that $\Delta K(x-\xi) = \delta(x-\xi)$).

(as part of this problem prove $u \in C^2(\mathbb{R}^n)$).

3. For $n \geq 3$ prove $\Delta K(\vec{x}) = 0 \ \forall \vec{x} \neq 0$. Hint: write $|x|^{2-n} = (|x|^2)^{\frac{2-n}{2}}$