Math 5440
Monday 11/1

§ 4.24 : Harmonic functions in disks.

By scaling and translation we consider the unit disk.

$$\text{D.P.} \begin{cases} \Delta u = 0 & x^2+y^2<1 \\ u = f & x^2+y^2=1 \end{cases}$$

polar coords allows us to separate variables.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

$$u = R(r)\Theta(\theta).$$

you did this in HW!

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R\Theta''(\theta) = 0$$

$\div \Theta R$, $* r^2$:

$$\frac{r^2R''+rR'}{R} = \frac{-\Theta''}{\Theta} = \text{const.} = \lambda$$

$\Theta(\theta)$ 2π-periodic $\Rightarrow$ $\lambda = n^2$, $n \in \mathbb{Z}^+$ $(n=0,1,2,\ldots)$, $\Theta(\theta) = \text{span}\{\cos n\theta, \sin n\theta\}$ $= \text{span}\{e^{in\theta}, e^{-in\theta}\}$.

$$r^2R''+rR'-n^2R = 0$$

$r = e^t$

$\tilde{R}(t) = R(e^t)$

$\tilde{R}' = R'e^t$

$\tilde{R}'' = R''e^{2t} + R'e^t$

$$\left.\begin{aligned} &\Rightarrow \tilde{R}''-n^2\tilde{R} = 0 \\ &\quad \tilde{R} = \text{span}\{e^{nt}, e^{-nt}\} \\ &\Rightarrow R(r) = \text{span}\{r^n, r^{-n}\}. \end{aligned}\right\} n \geq 1$$

$n=0$: $\tilde{R}'' = 0$

$\tilde{R} = c_1 + c_2 t$

$\Rightarrow R = c_1 + c_2 \ln r$

separated soltns : $\{1, \ln r, \{r^n\cos n\theta, r^{-n}\cos n\theta, r^n\sin n\theta, r^{-n}\sin n\theta\}_{n\in\mathbb{N}}$

<u>in the disk</u> eliminate $\ln r$, $r^{-n}\cos n\theta$, $r^{-n}\sin n\theta$

(<u>but</u> they'd be great in the exterior of a disk!

Formal soltn to Dirichlet problem :

Let f have Fourier series $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$

Then

$$\boxed{u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n\cos n\theta + \sum_{n=1}^{\infty} b_n r^n \sin n\theta}$$

is the formal soltn to Dirichlet problem.

- is $\Delta u = 0$ for $r<1$.

<u>ans</u> for $r \le 1-\delta$ $(\delta>0)$, series converges uniformly
and so does every series of iterated partial derivatives
$\Rightarrow$ $u(r,\theta)$ is C^∞ for $r<1$
and may be differentiated term by term
$\Rightarrow$ $\Delta u = 0$.

- Does $u=f$ on $\{r=1\}$?

yes, if the Fourier series for f converges pointwise.

- in fact, if f is continuous and piecewise C^1, then

$$S_N(f;\theta) \to f(\theta) \quad \text{uniformly}$$

maximum principle

$$\Rightarrow S_N(u;(r,\theta)) \to u(r,\theta) \quad \text{uniformly}.$$

$\Rightarrow$ $u(r,\theta)$ is uniform limit of continuous functions $\begin{array}{c} 0 \le r \le 1 \\ -\pi \le \theta < \pi \end{array}$
so is continuous, with
boundary values f.

Actually, the series sol'tn can be summed in closed form, leading to the <u>Poisson integral formula</u> (that we got earlier without checking all steps rigorously), which shows that the Dirichlet problem has solutions which extend continuously to the boundary for <u>f continuous</u>

It ends up being another "approximate identity" argument :)

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi + \sum_{n=1}^{\infty} r^n \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\big[\underbrace{\cos n\varphi \cos n\theta + \sin n\varphi \sin n\theta}_{\cos n(\theta-\varphi)}\big]\,d\varphi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\Big[1 + 2\sum_{n=1}^{\infty} r^n \cos n(\theta-\varphi)\Big]\,d\varphi$$

is a convolution
do usual COV

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\varphi)\Big(1 + 2\sum_{n=1}^{\infty} r^n \cos n\varphi\Big)\,d\varphi$$

$$P(r,\varphi) = \left(1 + 2\sum_{n=1}^{\infty} r^n \cos n\varphi\right) \qquad \text{Poisson kernel}$$

$$\sum_{n=1}^{\infty} r^n \cos n\varphi$$

$$= \operatorname{Re} \sum_{n=1}^{\infty} r^n e^{in\varphi}$$

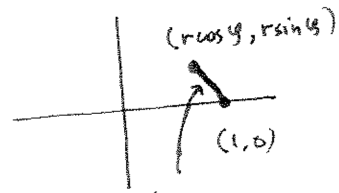
$$= \operatorname{Re} \frac{re^{i\varphi}}{1-re^{i\varphi}} \frac{1-re^{-i\varphi}}{1-re^{-i\varphi}}$$

$$= \frac{r\cos\varphi - r^2}{1-2r\cos\varphi + r^2}$$

so $$P(r,\varphi) = 1 + \frac{2r\cos\varphi - 2r^2}{1-2r\cos\varphi+r^2}$$

$$= \frac{1-r^2}{1-2r\cos\varphi + r^2}$$

$$= \frac{1-r^2}{(1-r\cos\varphi)^2 + (r\sin\varphi)^2}$$



denom is square of this length.

Approximate identity properties:

a) $\displaystyle\int_{-\pi}^{\pi} P(r,\varphi)\,d\varphi = 2\pi \qquad r<1$

b) $P(r,\varphi) > 0 \qquad r<1$

c) for $|\varphi| \geq \delta > 0$, $\quad 0 \leq P(r,\varphi) \leq \dfrac{1-r^2}{(\delta\sin\varphi)^2} \longrightarrow 0$ uniformly as $r \to 1^-$.

So,

Theorem: If f is continuous on ∂D, the harmonic function

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\varphi)\left(\frac{1-r^2}{(1-r\cos\varphi)^2+(r\sin\varphi)^2}\right)d\varphi$$

extends continuously to the closed unit disk $\{r \leq 1\}$, with boundary values f.

(proof Wed.).

Class exercises for Friday 11/5

(1) Let f be 2π-periodic, piecewise continuous and bounded.
Let $\varepsilon > 0$
Show $\exists\, g$ 2π-periodic, g continuous and piecewise C^1, so that

$$\|f-g\|_{L^2} = \left(\int_{-\pi}^{\pi} (f(x)-g(x))^2 \right)^{1/2} < \varepsilon.$$

(2) Let $0 < R_1 < R_2 < \infty$.
Use separation of variables to solve Dirichlet Problem for harmonic functions in an annulus:

$$DP \begin{cases} \Delta u = 0 & R_1 < \sqrt{x^2+y^2} < R_2 \\ u = f & \sqrt{x^2+y^2} = R_1 \\ u = g & \sqrt{x^2+y^2} = R_2 \end{cases}$$

- Find the formal Fourier series-like solution (i.e. separated variables sol'tn)
- Explain why the sol'tn is C^∞ for $R_1 < r < R_2$
  - and why it extends continuously to the closed annulus $R_1 \le r \le R_2$, with boundary values f, g in case f and g are both continuous.

(This explanation will likely involve Poisson integral formulas, scaling)