November 5, 2010
Math 5440
Homework #9

4.22.1:

$$\begin{cases} u_t - k u_{xx} = 0 & x \in (0,\pi),\ t>0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \sin^3 x \end{cases}$$

$u(x,0)=\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ $\underline{\text{is}}$ the Fourier series because projection onto a subspace is the identity on ~~that~~ subspace... no need to recompute Fourier coefficients!

$u(x,t) = X(x)T(t) \Rightarrow \frac{T'}{kT} - \frac{X''}{X} = 0$

$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = \lambda = -n^2 \qquad X(0)=X(\pi)=0 \qquad T(0)=1$

$X(x) = b_n \sin nx$

$b_n = \frac{2}{\pi}\int_0^\pi \sin^3 x \sin nx\,dx = \frac{1}{2\pi}\int_0^\pi (3\sin x - \sin 3x)\sin nx\,dx$

$= \frac{3}{2\pi}\int_0^\pi \sin x \sin nx\,dx - \frac{1}{2\pi}\int_0^\pi \sin 3x \sin nx\,dx$

$= \frac{3}{4\pi}\int_0^\pi (\cos(n-1)x - \cos(n+1)x)\,dx - \frac{1}{4\pi}\int_0^\pi (\cos(n-3)x - \cos(n+3)x)\,dx$

$= \frac{3}{4\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi - \frac{1}{4\pi}\left[\frac{\sin(n-3)x}{n-3} - \frac{\sin(n+3)x}{n+3}\right]_0^\pi$

$= 0$ except for $x = \pm 1, \pm 3$

$\lim_{n\to 1} \frac{3}{4\pi}\left[\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(0)}{n-1}\right] = \lim_{n\to 1}\frac{3}{4\pi}\left[\frac{\cos(n-1)\pi\ (\pi)}{1} - \frac{\cos(n-1)(0)\ (0)}{1}\right] = \frac{3}{4}$

$\lim_{n\to 3} \frac{-1}{4\pi}\left[\frac{\sin(n-3)\pi}{n-3} - \frac{\sin(0)}{n-3}\right] = \lim_{n\to 1}\frac{-1}{4\pi}\left[\frac{\cos(n-3)\pi\ (\pi)}{1} - \frac{\cos(n-3)(0)\ (0)}{1}\right] = -\frac{1}{4}$

$T' = -n^2 kT \qquad T(0)=1$

$T(t) = e^{-n^2kt} \qquad T(0)=1$

$$\boxed{u(x,0) = \frac{3}{4}\sin x\, e^{-kt} - \frac{1}{4}\sin 3x\, e^{-9kt}}$$

4.22.2.

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = x(\pi - x) \end{cases}$$

$X'' = -n^2 X \qquad X(0) = X(\pi) = 0$

$X(x) = b_n \sin nx$

$b_n = \underbrace{2\int_0^\pi x \sin nx\,dx}_{A} - \underbrace{\frac{2}{\pi}\int_0^\pi x^2 \sin nx\,dx}_{B}$

A: $u = x \quad dv = \sin nx\,dx$; $du = dx \quad v = -\cos nx \cdot \frac{1}{n}$

$A = \left[\frac{-x\cos nx}{n}\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx = \frac{2\pi}{n}$ n odd, $-\frac{2\pi}{n}$ n even

$A = \left[\frac{-x\cos nx}{n}\right]_0^\pi + \cancel{\frac{1}{n^2}\left[\sin nx\right]_0^\pi}$

B: $u = x^2 \quad dv = \sin nx$; $du = 2x\,dx \quad v = -\frac{1}{n}\cos nx$

$B = \frac{-1}{n}\left[x^2 \cos nx\right]_0^\pi + \frac{2}{n}\int_0^\pi x\cos nx\,dx \qquad u = x \quad dv = \cos nx\,dx \quad du = dx \quad v = \frac{\sin nx}{n}$

$B = \frac{-1}{n}\left[x^2\cos nx\right]_0^\pi + \cancel{\frac{2}{n^2}\left[x\sin nx\right]_0^\pi} - \frac{2}{n^2}\int_0^\pi \sin nx\,dx$

$= \frac{-1}{n}\left[x^2\cos nx\right]_0^\pi + \frac{2}{n^3}\left[\cos(nx)\right]_0^\pi = \frac{2\pi^2}{n} - \frac{4}{n^3}$ n odd $\qquad -\frac{2\pi^2}{n} + 0$ n even

$b_n = \frac{8}{\pi n^3}$ n odd $\qquad 0$ n even

$T' = -n^2 kT \quad T(0) = 1 \Rightarrow T(t) = e^{-n^2kt}$

$$\boxed{u(0,x) = \sum_{n=1}^{\infty} \frac{8}{\pi(2n-1)^3}\sin((2n-1)x)\, e^{-(2n-1)kt}}$$

4.23.3

$$\begin{cases} u_{xx}+u_{yy}=0 & x\in(0,\pi)\quad y\in(0,\pi) \\ u(\pi,y)=u(x,\pi)=u(0,y)=0 \\ u(x,0)=x^2(\pi-x) \end{cases}$$

$YX''+XY''=0 \qquad X''=-n^2X \quad X(0)=X(\pi)=0$

$X(x)=b_n\sin(nx)$

$Y''=n^2Y \quad Y(\pi)=0 \quad Y(0)=1$

$\Rightarrow Y(y)=\dfrac{\sinh(n(\pi-y))}{\sinh\pi}$

$b_n=\frac{2}{\pi}\int_0^{\pi} x^2(\pi-x)\sin nx\,dx \qquad u=\pi x^2-x^3 \quad dv=\sin nx\,dx \quad du=2\pi x-3x^2dx \quad v=-\frac{1}{n}\cos nx$

$=\frac{2}{\pi}\underbrace{\left[\left(-\frac{1}{n}\cos nx\right)(\pi x^2-x^3)\right]_0^{\pi}}_{0}+\frac{2}{\pi n}\int_0^{\pi}\cos nx\,(2\pi x-3x^2)\,dx \qquad u=2\pi x-3x^2 \quad dv=\cos nx\,dx \quad du=2\pi-6x\,dx \quad v=\frac{1}{n}\sin nx$

$=\cancel{\frac{2}{\pi n^2}\left[(2\pi x-3x^2)(\sin nx)\right]_0^{\pi}}^{0}-\frac{2}{\pi n^2}\int_0^{\pi}(2\pi-6x)\sin nx\,dx \qquad u=2\pi-6x \quad dv=\sin nx \quad du=-6 \quad v=-\frac{1}{n}\cos nx$

$=\frac{+2}{\pi n^3}\left[(2\pi-6x)(\cos nx)\right]_0^{\pi}+\frac{12}{\pi n^3}\int_0^{\pi}\cos nx\,dx$

$=\frac{2}{\pi n^3}\left[(2\pi-6x)(\cos nx)\right]_0^{\pi}+\cancel{\frac{12}{\pi n^4}\left[\sin nx\right]_0^{\pi}}^{0}$

$=\frac{2}{\pi n^3}\left[(2\pi-6\pi)\cos(n\pi)-2\pi\right]=\left[-4\pi\cos(n\pi)-2\pi\right]\frac{2}{\pi n^3}$

$=\left[-4\cancel{\pi}(-1)^n-2\cancel{\pi}\right]\frac{2}{\cancel{\pi}n^3}=\frac{-4}{n^3}\left(1+2(-1)^n\right)$

$$\boxed{u(x,y)=-4\sum_{n=1}^{\infty}\left(1+2(-1)^n\right)n^{-3}\sin nx\,\frac{\sinh(n(\pi-y))}{\sinh\pi}}$$

$|u(x,y)-S_N(x,y)|\le\left\{\frac{2}{\pi}\int_0^{\pi}f'^2dx-\sum_1^2 n^2b_n^2\right\}^{1/2}\left\{\frac{\pi^2}{6}-\sum_1^N\frac{1}{n^2}\right\}^{1/2}$

$\int_0^{\pi}(2\pi x-3x^2)^2\,dx=\frac{2\pi^5}{15} \qquad \sum_1^2 n^2b_n^2=\left(1\cdot(1-2)^2+4\cdot(1+2)^2/8^2\right)4^2=16+9$

$\sum_1^2\frac{1}{n^2}=1+\frac{1}{4}$

$|u(x,y)-S_N(x,y)|\le\left\{\frac{4\pi^4}{15}-16-9\right\}^{1/2}\left\{\frac{\pi^2}{6}-1-\frac{1}{4}\right\}^{1/2}\approx 0.621$

4.23.7

$$\begin{cases} \Delta u = 0 \quad x \in (0,\pi) \ \ y \in (0,\pi) \\ u(x,0) = u(x,\pi) = x^2 \\ u(0,y) = 0 \\ u(\pi,y) = \pi^2 \end{cases}$$

$v(x,y) = u(x,y) - [a + bx + cy + dxy]$

$v(0,0) = 0 - a = 0 \Rightarrow a = 0 \qquad v(\pi,0) = \pi^2 - b\pi = 0 \Rightarrow b = \pi$

$v(0,\pi) = 0 - c\pi = 0 \Rightarrow c = 0 \qquad v(\pi,\pi) = \pi^2 - \pi^2 + d\pi^2 = 0 \Rightarrow d = 0$

$\Delta v = 0$

$v(x,0) = x^2 - \pi x \qquad v(0,y) = 0$

$v(x,\pi) = x^2 - \pi x \qquad v(\pi,y) = 0$

$v_1(x,0) = x^2 - \pi x \qquad v_1(x,\pi) = 0 \qquad v_1 = X_1(x)\,Y_1(y)$

$-\frac{X_1''}{X_1} = \frac{Y_1''}{Y_1} = n^2 \qquad X_1(0) = X_1(\pi) = 0 \qquad Y(\pi) = 0 \quad Y(0) = 1$

$X_1(x) = b_n \sin(nx) \qquad Y_1(y) = \frac{\sinh n(\pi - y)}{\sinh n\pi}$

$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} (x^2 - \pi x)\sin nx\,dx = \frac{8}{\pi n^3}$ n odd (by 22.2)

$v_1(x,y) = \frac{8}{\pi}\sum_{n=1}^{\infty} (2n-1)^{-3} \frac{\sinh n(\pi - y)}{\sinh \pi} \sin nx$

$v_2(x,0) = 0 \qquad v_2(x,\pi) = x^2 - \pi x \qquad v_2 = X_2(x)\,Y_2(y)$

$-\frac{X_2''}{X_2} = \frac{Y_2''}{Y_2} = n^2 \qquad X_2(0) = X_2(\pi) = 0 \qquad Y(\pi) = 1 \quad Y(0) = 0$

$X_2 = b_n \sin(nx) \qquad Y_2(y) = \frac{\sinh ny}{\sinh n\pi}$

$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} (x^2 - \pi x)\sin nx\,dx = \frac{8}{\pi n^3}$ n odd

$v_2(x,y) = \frac{8}{\pi}\sum_{n=1}^{\infty} (2n-1)^{-3} \frac{\sinh ny}{\sinh \pi} \sin nx$

$$V = V_1 + V_2 = \frac{8}{\pi}\sum_{n=1}^{\infty} (2n-1)^{-3} \sin(2n-1)x \left[\frac{\sinh (2n-1)(\pi - y)}{\sinh (2n-1)\pi} + \frac{\sinh (2n-1)(y)}{\sinh (2n-1)\pi} \right]$$

4.24.1:

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ $r<$

$u(1,\theta) = \sin^3\theta$

$u(r,\theta) = R(r)\,\Theta(\theta) \Rightarrow \Theta R'' + \frac{1}{r}\Theta R' + \frac{1}{r^2}R\Theta''$

$R(1) = 1$

$n \geq 1$ $\quad C_1 r^n + \cancel{C_2 r^{-n}}$ → creates singularity at $r=0$

$R(1) = C_1 = 1 \qquad R(r) = r^n$

$n=0$ ~~not necessarry because this is a Dirichlet problem~~

so $a_0 = 0$

$\Theta(\theta) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \cancel{a_n} \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^3\theta \sin n\theta \, d\theta = \frac{2}{\pi}\int_{0}^{\pi} \sin^3\theta \sin n\theta \, d\theta$ Solved in problem 4.22.1 by trig identities

$b_1 = 3/4 \quad b_3 = -1/4 \quad b_n = 0 \ \forall n \neq 1,3$

$\boxed{u(r,\theta) = \cancel{\frac{3}{4}r - \frac{1}{4}r^3}}$

$\frac{3}{4} r \sin\theta - \frac{1}{4} r^3 \sin 3\theta$

4.24.8

$\nabla^2 u = 0 \quad r<1$ $\qquad \int_{-\pi}^{\pi} f(\theta)\,d\theta = 0$

$u_r(1,\theta) = f(\theta)$ $\qquad u(0,\theta) = 0$

θ - 2π periodic $\qquad \Theta$ - 2π - periodic

$u = R(r)\,\Theta(\theta)$

$n\geq 1 \quad R_n(r) = A r^n + B r^{-n} \qquad \Theta(\theta) = \{\cos n\theta, \sin n\theta\} \quad n \geq 1$

$n=0 \quad R_0(r) = c_1 + c_2 \ln r$

0 because of singularity

$u(r,\theta) = c_1 + \cancel{c_2 \ln r} \sum_{n=1}^{\infty} \bar{a}_n r^n \cos n\theta + \bar{b}_n r^n \sin n\theta$

$u(0,\theta) = 0 \Rightarrow c = 0$

$u_r(r,\theta) = \sum_{n=1}^{\infty} (n\bar{a}_n r^{n-1} \cos n\theta + n\bar{b}_n r^{n-1} \sin n\theta)$

$u_r(r,\theta) = \sum_{n=1}^{\infty} (a_n r^{n-1} \cos n\theta + b_n r^{n-1} \sin n\theta) \qquad a_n = n\bar{a}_n \quad b_n = n\bar{b}_n$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\,d\theta \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\,d\theta$

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} \left(\frac{a_n}{n} r^n \cos n\theta + \frac{b_n}{n} r^n \sin n\theta\right)}$$

if $\int_{-\pi}^{\pi} f(\theta)\,d\theta = c \neq 0$ then $a_0 = \frac{1}{\pi} c \Rightarrow \frac{a_0}{2} = \frac{c}{2\pi}$

then $u_r(r,\theta) = R_0'(r) : \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^{n-1} \cos n\theta + b_n r^{n-1} \sin n\theta$

So there must be some factor of $R_0 = c_1 + c_2 \ln r$ in $u(r,\theta)$. And $\nabla^2 u = 0$ so it must be a harmonic function $R(r) = c_2 \ln r$ is the only possibility and this creates a singularity at $r=0$ so there can be no solution

more generally, $\int_{\Omega} \Delta u = \int_{\partial\Omega} \nabla u \cdot \hat{n}$

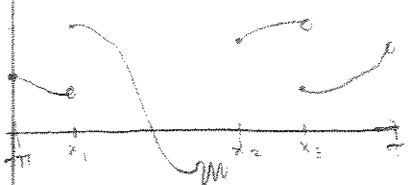
so if u satisfies $\Delta u = 0$ in Ω

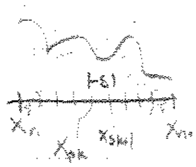
$\nabla u \cdot \hat{n} = f$ on $\partial\Omega$

then it must be that $\int_{\partial\Omega} f$ <u>must</u> equal 0

Class Exercise 1:

$\left(\int_{-\pi}^{\pi}|f(x)-g(x)|^2\right)^{1/2} < \varepsilon \Rightarrow \int_{-\pi}^{\pi}(f(x)-g(x))^2 < \varepsilon^2 = \varepsilon_0$





want $\int_{x_n+\varepsilon}^{x_{n+1}-\varepsilon}|f(x)-g(x)|^2 < \varepsilon_0/_2(x_1-x_2)$

let $g(x)$ be a p.w. linear function

$g(x) = f(x_{sk})\frac{(-x+x_{sk+1})}{\delta_1} + f(x_{sk+1})\frac{(x-x_{sk})}{\varepsilon_1}$ for $x \in [x_{sk}, x_{sk+1}]$

then $\exists\, \delta_1$ s.t. $|f(x)-g(x)| \le |f(x)-f(x_{sk})| + |f(x_{sk})-g(x)|$

$< 2\varepsilon_1 = \frac{\varepsilon_0^{1/2}}{\sqrt{2}}$ for $|x-x_{sk}|<\delta_1$

because f is p.w. cont so $|f(x)-f(x_{sk})|<\varepsilon_1$ and according to our definition of $g(x)$

$|f(x_{sk})-g(x)| < |f(x_{sk})-f(x)| < \varepsilon_1$

f is bdd so $|f| \le M$ $\forall x$ and g is close to f

so $|g| \le M$ $\forall x$ too so

Around a discontinuity define $g(x)$ as the following

$g(x) = f(x_n-\delta)\frac{(x-x_n+\delta)}{2\delta} + f(x_n+\delta)\frac{(x-x_n-\delta)}{2\delta}$ $x \in [x_n-\delta, x_n+\delta]$

then $|f(x)-g(x)| \le 2M$

so $\int_{x_n-\delta}^{x_n+\delta}|f(x)-g(x)|^2dx \le 4M^2\cdot 2\delta$ let $\delta = \frac{\varepsilon_0}{16M^2 n}$

then $\int_{-\pi}^{\pi}|f(x)-g(x)|^2 = \underbrace{\sum_{n=1}^{N}\int_{x_n-\delta}^{x_n+\delta}|f(x)-g(x)|^2dx}_{\mathrm{I}} + \underbrace{\sum_{n=1}^{N}\int_{x_n+\delta}^{x_{n+1}-\delta}|f(x)-g(x)|^2dx}_{\mathrm{II}} \le \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2}$

Class Exercise 2:

$$u(r,\theta) = R(r)\Theta(\theta)$$
$$R(A) = \tilde{a}_n \quad R(1) = \hat{a}_n$$

$n=0$ $R(r) = C_1 + C_2 \ln(r)$

$$C_1 + C_2 \ln A = a_0 \qquad \begin{bmatrix} 1 & \ln A \\ 1 & 0 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ \tilde{a}_0 \end{bmatrix} \Rightarrow \frac{-1}{\ln A}\begin{bmatrix} 0 & -\ln A \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_0 \\ \tilde{a}_0 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$
$$C_1 + 0 = \tilde{a}_0 \qquad \boxed{C_1 = \tilde{a}_0 \quad C_2 = \frac{a_0 - \tilde{a}_0}{\ln A}}$$

$n \geq 1$ $R(r) = C_3 r^n + C_4 r^{-n}$

$$C_3 A^n + C_4 A^{-n} = a_n \qquad \begin{bmatrix} A^n & A^{-n} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} a_n \\ \tilde{a}_n \end{bmatrix} \Rightarrow \frac{1}{A^n - A^{-n}}\begin{bmatrix} 1 & -A^{-n} \\ -1 & A^n \end{bmatrix}\begin{bmatrix} a_n \\ \tilde{a}_n \end{bmatrix} = \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}$$
$$C_3 + C_4 = \tilde{a}_n \qquad C_3 = \frac{a_n - A^{-n}\tilde{a}_n}{A^n - A^{-n}} \qquad C_4 = \frac{-a_n + A^n \tilde{a}_n}{A^n - A^{-n}}$$

$$u(r,\theta) = \frac{\tilde{a}_0}{2} + \frac{(a_0 - \tilde{a}_0)}{2}\frac{\ln r}{\ln A} + \sum_{n=1}^{\infty}\left[\left(\frac{A^n a_n - \tilde{a}_n}{A^{2n}-1}\right)r^n + \left(\frac{-a_n + A^{2n}\tilde{a}_n}{A^{2n}-1}\right)r^{-n}\right]\cos n\theta$$
$$+ \sum_{n=1}^{\infty}\left[\left(\frac{A^n b_n - \tilde{b}_n}{A^{2n}-1}\right)r^n + \left(\frac{-b_n + A^{2n}\tilde{b}_n}{A^{2n}-1}\right)r^{-n}\right]\sin n\theta$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta})\,d\bar{\theta} \qquad \tilde{a}_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\bar{\theta})\,d\bar{\theta}$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta})\cos n\bar{\theta}\,d\bar{\theta} \qquad \tilde{a}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\bar{\theta})\cos n\bar{\theta}\,d\bar{\theta}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta})\sin n\bar{\theta}\,d\bar{\theta} \qquad \tilde{b}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\bar{\theta})\sin n\bar{\theta}\,d\bar{\theta}$$

We have 2 types of terms $\left(\frac{A^n a_n - \tilde{a}_n}{A^{2n}-1}\right)r^n$ and $\left(\frac{-a_n + A^{2n}\tilde{a}_n}{A^{2n}-1}\right)r^{-n}$ the first terms all converge geometrically because $r<1$. The second type of term is less than $\left(\frac{A}{r}\right)^n$ $A<r$ so $\left(\frac{A}{r}\right)^n$ also converges geometrically. Any derivative with respect to r or θ of u will also converge geometrically.

this shows proposed soltn will satisfy $\Delta u = 0$, since you can pass the Δ operator thru summation

what abt boundary continuity/values?

Class exercise #2.

we know from previous HW & class that separated soltns for

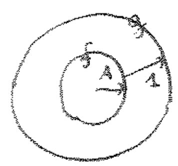
$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

are:

$n=0 \quad R_n\Theta_n = \text{span}\{\ln r, 1\}$.

$n \geq 1 \quad R_n\Theta_n = \text{span}\{r^n\cos n\theta, r^n \sin n\theta, r^{-n}\cos n\theta, r^{-n}\sin n\theta\}$.

by scaling annulus we may assume domain is $A \leq r \leq 1$



$$\Delta u = 0, \quad A < r < 1$$
$$u = f \quad r = A$$
$$u = g \quad r = 1.$$

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

$$g \sim \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta$$

interpolate each term with $R_n(r)$.

$n=0$
$$\begin{bmatrix} 1 & \ln A \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_0/2 \\ \tilde{a}_0/2 \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{\ln A}\begin{bmatrix} 0 & -\ln A \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_0/2 \\ \tilde{a}_0/2 \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{a}_0/2 \\ \frac{1}{\ln A}\left(\frac{a_0}{2} - \frac{\tilde{a}_0}{2}\right) \end{bmatrix}$$

$n \geq 1$:
$$\begin{bmatrix} A^n & A^{-n} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_n \\ \tilde{a}_n \end{bmatrix} \quad \left(\text{or } \begin{bmatrix} b_n \\ \tilde{b}_n \end{bmatrix}\right)$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{A^n - A^{-n}}\begin{bmatrix} 1 & -A^{-n} \\ -1 & A^n \end{bmatrix}\begin{bmatrix} a_n \\ \tilde{a}_n \end{bmatrix} = \frac{1}{A^n - A^{-n}}\begin{bmatrix} a_n - A^{-n}\tilde{a}_n \\ -a_n + A^n\tilde{a}_n \end{bmatrix} = \frac{1}{A^{2n}-1}\begin{bmatrix} a_n A^n - \tilde{a}_n \\ -a_n A^n + A^{2n}\tilde{a}_n \end{bmatrix}$$

So,

$$\boxed{\begin{aligned} u(r,\theta) = \tilde{a}_0/2 + \frac{\ln r}{\ln A}\left(\frac{a_0}{2} - \frac{\tilde{a}_0}{2}\right) &+ \sum_{n=1}^{\infty}\left[r^n\left(\frac{\tilde{a}_n - a_n A^n}{1-A^{2n}}\right) + r^{-n}\left(\frac{a_n A^n - \tilde{a}_n A^{2n}}{1-A^{2n}}\right)\right]\cos n\theta \\ &+ \sum_{n=1}^{\infty}\left[r^n\left(\frac{\tilde{b}_n - b_n A^n}{1-A^{2n}}\right) + r^{-n}\left(\frac{b_n A^n - \tilde{b}_n A^{2n}}{1-A^{2n}}\right)\right]\sin n\theta \end{aligned}}$$

Convergence: coeffs of r^n, r^{-n} involve Fourier coeffs $a_n, \tilde{a}_n, b_n, \tilde{b}_n$ which are all bounded, as well as $\cos n\theta$, $\sin n\theta$.

Thus terms are bounded in abs. value by fixed multiples of

$$\frac{r^n A^n}{|A^{2n}-1|} \le C\, r^n A^n \qquad (\text{since } A^{2n} \to 0).$$

$$\frac{r^n}{|A^{2n}-1|} \le C r^n$$

$$\frac{r^{-n} A^n}{|A^{2n}-1|} \le C\left(\frac{A}{r}\right)^n$$

$$\frac{r^{-n} A^{2n}}{|A^{2n}-1|} \le C\left(\frac{A}{r}\right)^n A^n$$

all terms decrease geometrically for

$$A+\delta \le r \le 1-\delta$$

also, after iterated r & θ derivs, ~~terms are~~ series are still uniformly convergent for

$$A+\delta \le r \le 1-\delta$$

$\rightarrow$ u is C^∞, $\partial_r\partial_r + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta\partial_\theta$ passes thru the sum, and <u>u is harmonic</u>.

Does u extend continuously to the closed annulus $A \le r \le 1$, with boundary values f, g?

<u>Answer 1</u>: If f, g are cont. & piecewise C^1 (or $\int_{-\pi}^{\pi}(f')^2 d\theta$, $\int_{-\pi}^{\pi}(g')^2 d\theta < \infty$), their Fourier series converge uniformly, so by the maximum principle the partial sums for u converge uniformly at the same rate, so the limit function is continuous for $A \le r \le 1$, with boundary values f, g.

<u>Answer 2</u>: If you only assume f, g continuous you can use two Poisson integral formulas to replace some of the terms (one for f, one for g) and what's left will converge uniformly on the closed annulus, which proves the stronger result that u extends continuously to the closed annulus with bdry values f, g.

This is a mess to do, and it won't be graded.
For my penance for assigning the problem,
details are on next page

using Poisson,

$$\frac{\tilde a_0}{2} + \sum_{n=1}^{\infty} \tilde a_n r^n \cos n\theta + \tilde b_n r^n \sin n\theta = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{(1-r\cos\varphi)^2+(r\sin\varphi)^2}\, g(\theta-\varphi)\,d\varphi}_{\text{and this fcn extends continuously to } r=1, \text{ with boundary values } g.} \qquad r<1$$

call this $I_g(r,\theta)$

~~replacing~~ note, if $r > A$

then $\frac{1}{r} < \frac{1}{A}$

$\frac{A}{r} < 1$

so

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{A}{r}\right)^n \cos n\theta + b_n \left(\frac{A}{r}\right)^n \sin n\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-(A/r)^2}{(1-A/r\cos\varphi)^2+(A/r\sin\varphi)^2}\, g(\theta-\varphi)\,d\varphi$$

for $A/r < 1$ $(r > A)$

and extends continuously to f for $r=A$.

call this $I_f(r,\theta)$.

note $\displaystyle \frac{1}{1-A^{2n}} = \frac{1-A^{2n}}{1-A^{2n}} + \frac{A^{2n}}{1-A^{2n}} = 1 + \frac{A^{2n}}{1-A^{2n}}$

so we may rewrite page 1

$$u(r,\theta) = \frac{\tilde a_0}{2} + \frac{\ln r}{\ln A}\left(\frac{a_0}{2} - \frac{\tilde a_0}{2}\right) + \sum_{n=1}^{\infty}\left(1+\frac{A^{2n}}{1-A^{2n}}\right)\left[r^n \tilde a_n - a_n r^n A^n + \left(\frac{A}{r}\right)^n a_n - r^{-n}A^{2n}\tilde a_n\right]\cos n\theta$$

$$+ \sum_{n=1}^{\infty}\left(1+\frac{A^{2n}}{1-A^{2n}}\right)\left[r^n \tilde b_n - b_n r^n A^n + \left(\frac{B}{r}\right)^n b_n - r^{-n}A^{2n}\tilde b_n\right]\sin n\theta$$

these sum to $I_g(r,\theta)$

these sum to $I_f(r,\theta)$

and all other terms converge uniformly on the closed annulus $a \le r \le 1$

Note to those who've had $\mathbb{C}$-analysis:
I thought this would work out more easily than it did, since it's the real analog of the Cauchy Integral formula for an annulus, just as the Poisson Int. formula is for the disk.

whew.