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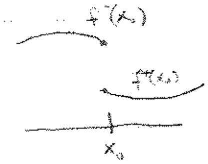
Math 5440
Homework #8

Problem 1:



$f: \mathbb{R} \to \mathbb{R}$, 2π periodic, bounded,
p.w. C^1 but has jump discontinuity at x_0.

Show that $\lim_{N\to\infty} S_N(f, x_0) = \frac{1}{2}\left(f^-(x_0) + f^+(x_0)\right)$

$$S_N(f, x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0 - t) D_N(t)\,dt$$

$$D_N(t) = \frac{\sin((N+\frac{1}{2})t)}{\sin(t/2)} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{0} D_N(t)\,dt = \frac{1}{2\pi}\int_{0}^{\pi} D_N(t)\,dt = \frac{1}{2}$$

$$\frac{1}{2}(f^- + f^+) - S_N(f, x_0) = \frac{1}{2}(f^- + f^+) - \frac{1}{2\pi}\int_{-\pi}^{0} f(x_0 - t) D_N(t)\,dt - \frac{1}{2\pi}\int_{0}^{\pi} f(x_0 - t) D_N(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} f^+ D_N(t)\,dt + \frac{1}{2\pi}\int_{0}^{\pi} f^- D_N(t)\,dt - \frac{1}{2\pi}\int_{-\pi}^{0} f(x_0-t) D_N(t)\,dt - \frac{1}{2\pi}\int_{0}^{\pi} f(x_0-t) D_N(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} \left(f^+ - f(x_0 - t)\right) D_N(t)\,dt + \frac{1}{2\pi}\int_{0}^{\pi} \left(f^- - f(x_0-t)\right) D_N(t)\,dt$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{0} \underbrace{\frac{(f^+ - f(x_0-t))}{\sin(t/2)}}_{=: g^+(t)} \sin((N+\tfrac{1}{2})t)\,dt}_{1} + \underbrace{\frac{1}{2\pi}\int_{0}^{\pi} \underbrace{\frac{(f^- - f(x_0-t))}{\sin(t/2)}}_{=: g^-(t)} \sin((N+\tfrac{1}{2})t)\,dt}_{2}$$

1) Define $\langle f, g\rangle = \frac{2}{\pi}\int_{-\pi}^{0} f(t)\, g(t)\,dt$

then $V = \{u_1, u_2, \dots, u_N, \dots\}$ $u_i \perp u_j \;\forall i \neq j$

then $\text{proj}_{V_n} f = \sum_{j=1}^{n} \langle f, \frac{u_j}{\|u_j\|}\rangle \frac{u_j}{\|u_j\|}$ if $\forall u_j$ are orthogonal

and $\|u_i\| = \|u_j\| = 1 \;\forall i,j$ then Bessels inequality holds:

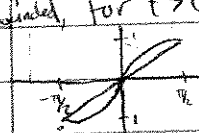
$$\|\text{proj}_{V_n} f\|^2 = \sum_{j=1}^{n} \langle f, u_j\rangle^2 \le \|f\|^2$$

This means that if g^+ is bounded and $\{\sin((N+\frac{1}{2})t)\}_{n=1}^{\infty}$ is an orthonormal set, then $\lim_{N\to\infty} (1)^2 = 0$

$\Rightarrow \lim_{N\to\infty} (1) = 0.$

g^+ is bounded: $|g^+| = \left|\frac{(f^+ - f(x_0-t))}{t}\, 2\, \frac{\frac{1}{2}t}{\sin t/2}\right| = \frac{|f^+ - f(x_0-t)|}{|t|}\,\frac{|\frac{1}{2}t|}{|\sin t/2|}\, 2$

because f is C^1 bounded, for $t > 0$ then $\exists C$ s.t. $|f^+ - f(x_0 - t)| \le C|t| \Rightarrow \frac{|f^+ - f(x_0-t)|}{|t|} \le C.$



$\left|\frac{\sin\theta}{\theta}\right| \ge \frac{2}{\pi} \Rightarrow \left|\frac{\theta}{\sin\theta}\right| \le \frac{\pi}{2}$

So $|g^+| \le C\pi = \tilde{C} \;\forall t$

$\{\sin((N+\frac{1}{2})t)\}_{n=1}^{\infty}$ is an orthonormal set.

$\|u_N\| = \int_{-\pi}^{0} \sin^2((N+\frac{1}{2})t)\,dt = \frac{\pi}{2} \;\forall N$

Show $\langle \sin((M+\tfrac{1}{2})t), \sin((N+\tfrac{1}{2})t) \rangle = 0$ for $M \neq N$, assume $M = N+k$

$= \frac{2}{\pi}\int_{-\pi}^{0} \sin((M+\tfrac{1}{2})t)\sin((N+\tfrac{1}{2})t)\,dt \qquad \sin\alpha\sin\beta = [\cos(\alpha-\beta) - \cos(\alpha+\beta)]\tfrac{1}{2}$

$= \frac{2}{\pi}\int_{-\pi}^{0} \cos((M-N)t) - \cos((M+N+1)t)\,dt$

$= \frac{2}{\pi}\int_{-\pi}^{0} \cos(kt) - \cos((2N+2k+1)t)\,dt$

$= \frac{-2}{\pi}[\sin(k\pi) - \sin(0) - \sin((2N+2k+1)(-\pi)) - \sin(0)] = 0 \quad \forall N, k$

So $\{\sin((N+\tfrac{1}{2})t)\}_{n=1}^{\infty}$ is an orthonormal set.

So $\lim_{N\to\infty} \langle g^{+}, \sin((N+\tfrac{1}{2})t) \rangle = 0$

$\Rightarrow \lim_{N\to\infty} \frac{1}{4}\left[\frac{2}{\pi}\int_{-\pi}^{0} \frac{f^{+} - f(x_0-t)}{\sin(t/2)} \sin((N+\tfrac{1}{2})t)\,dt\right] = 0.$

The same result can be shown for ② in a similar manner, defining the inner product $\langle f, g \rangle = \frac{2}{\pi}\int_{0}^{\pi} fg\,dt$

$\lim_{N\to\infty} \frac{1}{4}\left[\frac{2}{\pi}\int_{0}^{\pi} \frac{f^{-} - f(x_0-t)}{\sin(t/2)} \sin((N+\tfrac{1}{2})t)\,dt\right] = 0$

$\lim_{N\to\infty} \frac{1}{2}(f^{+} + f^{-}) - S_N(f, x_0) = \lim_{N\to\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{0} \frac{f^{+} - f(x_0-t)}{\sin(t/2)} \sin(N+\tfrac{1}{2})t\,dt + \frac{1}{2\pi}\int_{0}^{\pi} \frac{f^{-} - f(x_0-t)}{\sin(t/2)} \sin(N+\tfrac{1}{2})t\,dt\right]$

$= 0.$

So $\boxed{\lim_{N\to\infty} S_N(f, x_0) = \frac{1}{2}(f^{+} + f^{-})}$

Problem 2:

$$K_N = \frac{1}{N+1}\sum_{n=0}^{N} D_n = \frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-N}^{N} e^{ikt}$$

$$= \frac{1}{N+1}\sum_{n=0}^{N}\left[\sum_{k=-N}^{-1} e^{ikt} + 1 + \sum_{k=1}^{N} e^{ikt}\right]$$

$$= \frac{1}{N+1}\sum_{n=0}^{N}\left[1 + 2\,\mathrm{Re}\left(\sum_{k=1}^{N} e^{ikt}\right)\right]$$

$$\sum_{k=1}^{N} e^{ikt} = e^{it}\left(\frac{1-e^{int}}{1-e^{it}}\right)$$

$$= \frac{(e^{it} - e^{i(n+1)t})(1-e^{-it})}{2-2\cos t}$$

$$= \frac{e^{it} - e^{i(n+1)t} - 1 + e^{int}}{2-2\cos t}$$

$$\mathrm{Re}\left(\sum_{k=1}^{N} e^{ikt}\right) = \frac{\cos(t) - \cos((n+1)t) - 1 + \cos(nt)}{2(1-\cos t)}$$

$$= \frac{1}{N+1}\sum_{n=0}^{N}\left[1 + \frac{\cos t - 1}{1-\cos t} + \frac{\cos(nt) - \cos((n+1)t)}{(1-\cos t)}\right]$$

$$= \frac{1}{N+1}\frac{1}{1-\cos t}\left[\cos(0) - \cos(t) + \cos(t) - \cos(2t) + \ldots - \cos((N+1)t)\right]$$

$$= \boxed{\frac{1}{N+1}\frac{1-\cos((N+1)t)}{1-\cos t}}$$

a. Show that $K_N \geq 0 \;\forall t$

for all $t \neq 0$ then $\cos((N+1)t) < 1$, $\cos t < 1$

so $K_N = \frac{1}{N+1}\frac{1-\cos((N+1)t)}{1-\cos t} > \frac{1}{N+1} > 0$.

for $t = 0$:

$$\lim_{t\to 0} K_N = \lim_{t\to 0}\frac{1}{N+1}\frac{1-\cos((N+1)t)}{1-\cos t} = \lim_{t\to 0}\frac{1}{\cancel{N+1}}\frac{\cancel{(N+1)}\sin((N+1)t)}{\sin t}$$

$$= \lim_{t\to 0}(N+1)\frac{\cos((N+1)t)}{\cos t} = N+1 \geq 0.$$

So $K_N \geq 0 \;\forall t$

b. $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,dt$

$$= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N = \frac{1}{N+1}\sum_{n=0}^{N} 1 = \frac{N+1}{N+1} = 1$$

c. $K_N(x) \leq \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$ if $0 < \delta \leq |x| \leq \pi$

$\cos(x)$ is decreasing as $|x| \to \pi$

so $\cos(x) \leq \cos(\delta)$ for $|x| \geq \delta$.

so $1 - \cos\delta \leq 1 - \cos x$

and $1 - \cos(kx) \leq 2 \;\forall x$ so $1 - \cos((N+1)x) \leq 2$

So $K_N(x) = \frac{1}{N+1}\frac{1-\cos((N+1)x)}{1-\cos x} \leq \frac{1}{N+1}\frac{2}{1-\cos\delta}$

Prove $\sigma_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt$

$$\sigma_N = \frac{1}{N+1}\sum_{n=0}^{N} S_n$$

$$\sigma_N = \frac{1}{N+1}\sum_{n=0}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,D_n(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,\frac{1}{N+1}\sum_{n=0}^{N} D_n(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,K_N(t)\,dt$$

$$f(x)-\sigma_N(f;x) = f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,K_N(t)\,dt$$

$$= f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(f(x)-f(x-t)\big)K_N(t)\,dt$$

$$|f(x)-\sigma_N(f;x)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(f(x)-f(x-t)\big)\,K_N(t)\,dt\right|$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-f(x-t)|\,|K_N(t)|\,|dt|$$

$$= \underbrace{\frac{1}{2\pi}\int_{|t|<\delta}|f(x)-f(x-t)|\,K_N(t)\,dt}_{I_1} + \underbrace{\frac{1}{2\pi}\int_{\pi\ge|x|>\delta}|f(x)-f(x-t)|\,K_N(t)\,dt}_{I_2}$$

$$I_1 = \frac{1}{2\pi}\int_{|t|<\delta}|f(x)-f(x-t)|\,K_N(t)\,dt$$

$|f(x)-f(x-t)|\le\varepsilon$ for some δ, because f is continuous.

$$\Rightarrow I_1 \le \frac{1}{2\pi}\int_{|t|<\delta}\varepsilon\,K_N(t)\,dt = \frac{\varepsilon}{2\pi}\int_{|t|<\delta}K_N(t)\,dt \le \frac{\varepsilon}{2\pi}$$

because $\int_{-\pi}^{\pi}K_N(t)\,dt = 1$ and $K_N(t)\ge 0\ \forall t$.

$$I_1 = \frac{1}{2\pi}\int_{\pi>|t|>\delta}|f(x)-f(x-t)|\,K_N(t)\,dt$$

$|f(x)-f(x-t)|\le M$ because f is continuous.

$K_N(t)\le\frac{1}{N+1}\left(\frac{2}{1-\cos\delta}\right)$; Defin $C_\delta = \frac{1}{2-\cos\delta}$ Then for a fixed δ,

$K_N(t)\le\frac{1}{N+1}C_\delta$, Choose N big enough and

$K_N(t)\le\varepsilon^*$

Let $\varepsilon_0>0$. Then choose $\delta>0$ s.t. $|f(x)-f(x-t)|\le\varepsilon_0/2$ for $|t|<\delta$. Then $I_1 = \frac{1}{2\pi}\int_{|t|<\delta}|f(x)-f(x-t)|\,K(t)\,dt \le \varepsilon_0/2$

Also $I_2 = \int_{\pi\ge|t|\ge\delta}|f(x)-f(x-t)|K_N(t)\,dt \le \frac{MC_\delta}{N+1}$ So, for fixed δ,

$\exists N$ s.t. for $n\ge N$ $\frac{MC_\delta}{n+1}<\varepsilon_0/2$.

Then $I_1+I_2 = |f(x)-\sigma_N(f;x)|\le\varepsilon_0\ \forall x$ ∎

Problem 3: Done in class on October 27, Completion of this problem not required for this assignment.

Problem 4:

Dirichlet Problem:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi}{\ell}ct\right) + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{n\pi}{\ell}ct\right)$$

$$= \sum_{n=1}^{\infty} \frac{b_n}{2}\left[\sin\left(\frac{n\pi}{\ell}(x+ct)\right) + \sin\left(\frac{n\pi}{\ell}(x-ct)\right)\right] + \sum_{n=1}^{\infty}\frac{c_n}{2}\left[\cos\left(\frac{n\pi}{\ell}(x-ct)\right) - \cos\left(\frac{n\pi}{\ell}(x+ct)\right)\right]$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}(x+ct)\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}(x-ct)\right)\right]$$

$$+ \frac{1}{2c}\left[\sum_{n=1}^{\infty} c_n \cdot \frac{n\pi c}{\ell}\int_{x-ct}^{x+ct} \sin\left(\frac{n\pi}{\ell}\bar{x}\right)d\bar{x}\right]$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}(x+ct)\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}(x-ct)\right)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty} c_n \cdot \frac{n\pi c}{\ell}\sin\left(\frac{n\pi}{\ell}\bar{x}\right)d\bar{x}$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$$

for $f(\bar{x}) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}\bar{x}\right)$ and $g(\bar{x}) = \sum_{n=1}^{\infty} c_n \frac{n\pi c}{\ell}\sin\left(\frac{n\pi}{\ell}\bar{x}\right)$

both $f(\bar{x})$ and $g(\bar{x})$ have odd reflections about $\bar{x}=0$ and $\bar{x}=\ell$.

Neumann Problem:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi}{\ell}ct\right) + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{n\pi}{\ell}ct\right)$$

$$= \sum_{n=1}^{\infty}\frac{b_n}{2}\left[\cos\left(\frac{n\pi}{\ell}(x+ct)\right) + \cos\left(\frac{n\pi}{\ell}(x-ct)\right)\right] + \sum_{n=1}^{\infty}\frac{c_n}{2}\left[\sin\left(\frac{n\pi}{\ell}(x+ct)\right) - \sin\left(\frac{n\pi}{\ell}(x-ct)\right)\right]$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{\ell}(x+ct)\right) + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{\ell}(x-ct)\right)\right]$$

$$+ \frac{1}{2c}\left[\sum_{n=1}^{\infty} c_n \cdot \frac{n\pi c}{\ell}\int_{x-ct}^{x+ct}\cos\left(\frac{n\pi}{\ell}\bar{x}\right)d\bar{x}\right]$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{\ell}(x+ct)\right) + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{\ell}(x-ct)\right)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty} c_n \frac{n\pi c}{\ell}\sin\left(\frac{n\pi}{\ell}\bar{x}\right)d\bar{x}$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$$

for $f(\bar{x}) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{\ell}\bar{x}\right)$ and $g(\bar{x}) = \sum_{n=1}^{\infty} c_n \frac{n\pi}{\ell}c \sin\left(\frac{n\pi}{\ell}\bar{x}\right)$

both $f(\bar{x})$ and $g(\bar{x})$ have even reflections about $\bar{x}=0$ and $\bar{x}=\ell$.

Problem 5:

4.23.1:

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0<x<B \quad 0<y<A \\ u(0,y) = u(B,y) = u(x,A) = 0 \\ u(x,0) = f(x) \end{cases}$$

$u(x,t) = X(x)Y(y) \qquad YX'' + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

$X(0) = X(B) = \quad \lambda < 0 \Rightarrow \lambda = -k^2$

$X(x) = a\sin(kx) + b\cos(kx)$

$X(0) = b\cos(k(x)) = 0 \Rightarrow b = 0$

$X(B) = a(\sin(kB)) = a\sin(0) \Rightarrow kB = n\pi \Rightarrow k = \frac{n\pi}{B}$

$X(x) = a\sin(\frac{n\pi}{B}x)$

$Y'' = \frac{n^2\pi^2}{B^2}Y, \quad Y(A) = 0 \Rightarrow Y = c\sinh(\frac{n\pi}{B}(\phi - y))$

$Y(A) = c\sinh(\frac{n\pi}{B}(\phi - A)) = 0 \quad \phi - A = 0 \quad \phi = A$

$Y(y) = c\sin(\frac{n\pi}{B}(A-y))$

$Y(0) = 1 \Rightarrow c\sin(\frac{n\pi}{B}A) \Rightarrow c = \frac{1}{\sin(\frac{n\pi}{B}A)}$

$a_n = \frac{2}{B}\int_0^B \sin(\frac{n\pi}{B}x)\cdot f(x)\,dx$

$$\boxed{\begin{gathered} u(x,y) = \sum_{n=1}^{\infty} a_n \frac{\sinh(\frac{n\pi}{B}(A-y))}{\sinh(\frac{n\pi}{B}A)} \sin(\frac{n\pi}{B}x) \\ a_n = \frac{2}{B}\int_0^B f(x)\sin(\frac{n\pi}{B}x)\,dx \end{gathered}}$$

4.23.2.

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0<x<\pi \quad 0<y<A \\ u(0,y) = g(y) \\ u(\pi,y) = u(x,0) = u(x,A) = 0 \end{cases}$$

$\lambda = \frac{X''}{X} = -\frac{Y''}{Y} \quad X(0) = 1, \ X(\pi) = 0$

$Y(0) = Y(A) = 0$

Just like above we want a solution of the form $u(x,y) = \sum_{n=1}^{\infty} a_n \sinh(\sqrt{\lambda}(\phi - x)) \cdot c_n \sin(\sqrt{\lambda}(y))$

$Y(A) = c\sin(\sqrt{\lambda}A) = c\sin(0) \Rightarrow \sqrt{\lambda} = \frac{n\pi}{A}$

$c_n = \frac{2}{A}\int_0^A g(y)\sin(\frac{n\pi}{A}y)\,dy$

$X(\pi) = a_n\sinh(\sqrt{\lambda}(\phi - \pi)) = 0 \Rightarrow \phi - \pi = 0 \Rightarrow \phi = \pi$

$X(0) = a_n\sin(\frac{n\pi}{A}(\pi - 0)) = 1 \Rightarrow a_n = \frac{1}{\sinh(\frac{n\pi^2}{A})}$

$$\boxed{\begin{gathered} u(x,y) = \sum_{n=1}^{\infty} c_n \frac{\sinh(\frac{n\pi}{A}(\pi - x))}{\sinh(\frac{n\pi^2}{A})} \sin(\frac{n\pi}{A}y) \\ c_n = \frac{2}{A}\int_0^A g(y)\sin(\frac{n\pi}{A}y)\,dy \end{gathered}}$$