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Math 5440
Homework # 7.

1.14, 1a.

$$\begin{cases} u_{tt} - u_{xx} - u = 0 & 0<x<1 \quad t>0 \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

$u(x,t) = X(x)T(t) \Rightarrow XT'' - TX'' - TX = 0$

$\frac{T''}{T} - 1 = \frac{X''}{X} = \lambda \qquad \lambda = -n^2\pi^2 \qquad \lambda + 1 = -(n^2\pi^2 - 1)$

$\left.\begin{array}{l} X'' = \lambda X \\ X(0) = X(1) = 0 \end{array}\right| \Rightarrow X(x) = \sin(n\pi x) \leftarrow$ (done in class oct 6)

$\left.\begin{array}{l} T'' = T(\lambda+1) \\ T'(0) = 0 \end{array}\right| \Rightarrow T(t) = \cos(\sqrt{n^2\pi^2 - 1}\; t)$

$$\boxed{u(x,t) = \sin(n\pi x)\cos(\sqrt{n^2\pi^2-1}\; t) \qquad n \in \mathbb{N}}$$

d. $$\begin{cases} u_t - t^2 u_{xx} - u = 0 & 0<x<1 \quad t>0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

$u(x,t) = X(x)T(t) \Rightarrow XT' - t^2 XT'' - XT = 0$

$\left(\frac{T'}{T} - 1\right)\frac{1}{t^2} = \frac{X''}{X} = \lambda \Rightarrow X(x) = \sin(n\pi x) \qquad \lambda = -n^2\pi^2$

$T' = T(t^2\lambda + 1) \Rightarrow T(t) = e^{\left(\frac{t^3}{3}\lambda + t\right)} \Rightarrow T(t) = e^{t - \frac{n^2\pi^2 t^3}{3}}$

$$\boxed{u(x,t) = \sin(n\pi x)\, e^{\left(t - \frac{n^2\pi^2}{3} t^3\right)}}$$

4.14.2.

$\Delta u = 0 \qquad r = [x^2+y^2]^{1/2} \qquad \Theta = \tan^{-1}(y/x)$

$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

$= \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

$= \frac{1}{r}u_r + u_{rr} + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad U = R(r)\,\Theta(\theta)$

$= \frac{1}{r}\Theta R' + \Theta R'' + \frac{1}{r^2}R\Theta'' = 0 \qquad \lambda < 0$

$\frac{rR' + r^2R''}{R} = -\frac{\Theta''}{\Theta} = \lambda$

$\Theta'' = -\lambda\Theta \Rightarrow \Theta = A\cos(\sqrt{\lambda}\,\theta + \varphi)$ or $\Theta = c_1 e^{\sqrt{-\lambda}\theta} + c_2 e^{-\sqrt{-\lambda}\theta}$

but this soln fails to be 2π-periodic.

$\Theta(0) = \Theta(2\pi) \Rightarrow A\cos(\varphi) = A\cos(\sqrt{\lambda}\,2\pi + \varphi)$

$\Rightarrow 2\pi\sqrt{\lambda} = 2\pi n \Rightarrow \lambda = n^2$

$\Theta(\theta) = A\cos(n\theta + \varphi)$

$\frac{rR' + r^2R''}{R} = n^2 \Rightarrow R''r^2 + R'r - n^2R = 0$

$r = e^t \quad t = \ln(r) \qquad \frac{dt}{dr} = \frac{1}{r} \qquad \frac{d^2t}{dr^2} = \frac{-1}{r^2}$

$R(r) = T(t) \qquad R'(r) = T'\frac{dt}{dr} = T'\cdot\frac{1}{r}$

$R''(r) = T''\left(\frac{dt}{dr}\right)^2 + T'\frac{d^2t}{dr^2} = (T'' - T')\frac{1}{r^2}$

$R''r^2 + R'r - n^2R = 0 = T'' - \cancel{T'} + \cancel{T'} - n^2T$

$T'' = n^2T \Rightarrow T(t) = e^{\pm nt} = (e^t)^{\pm n} = r^{\pm n} \Rightarrow R(r) = Br^n + Cr^{-n}$

$\boxed{u(r,\theta) = \cos(n\theta + \varphi)(Ar^n + Br^{-n})}$

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Class Exercises.

1\. a. $(\mathbb{C}^n, \langle\ ,\ \rangle)$ $\quad \langle z, \omega \rangle = \sum_{j=1}^{n} z_j \overline{\omega_j}$

i. $\langle f+g, h\rangle = \sum_{j=1}^{n} (f+g)_j \overline{h_j} = \sum (f_j + g_j)\overline{h_j} = \sum f_j \overline{h_j} + \sum g_j \overline{h_j} = \langle f,h\rangle + \langle g,h\rangle$

$\langle f, g+h\rangle = \sum_{j=1}^{n} f_j \overline{(g+h)_j} = \sum f_j(\overline{g_j} + \overline{h_j}) = \sum f_j \overline{g_j} + \sum f_j \overline{h_j} = \langle f,g\rangle + \langle f,h\rangle$

ii. $\langle cf, h\rangle = \sum_{j=1}^{n} cf_j \overline{h_j} = c\sum_{j=1}^{n} f_j \overline{h_j} = c\langle f,h\rangle$

$\langle f, ch\rangle = \sum_{j=1}^{n} f_j \overline{(ch)_j} = \sum_{j=1}^{n} f_j \overline{c}\,\overline{h_j} = \overline{c}\sum_{j=1}^{n} f_j \overline{h_j} = \overline{c}\langle f,h\rangle$

iii. $\overline{\langle g,f\rangle} = \overline{\sum_{j=1}^{n} g_j \overline{f_j}} = \sum_{j=1}^{n} \overline{g_j} f_j = \langle f,g\rangle$

iv. $\langle f,f\rangle = \sum_{j=1}^{n} f_j \overline{f_j} = \sum_{j=1}^{n} |f_j|^2 \quad |f_j|^2 \geq 0\ \forall_j$

$\therefore \sum_{j=1}^{n} |f_j|^2 \geq 0$ if $\sum_{j=1}^{n} |f_j|^2 = 0 \Rightarrow f_j = 0\ \forall_j \Rightarrow f = 0$

if $f = 0$ then $\langle f,f\rangle = \sum_{j=1}^{n} (0)(0) = 0$.

b. $\vec{x} + i\vec{y} \perp \vec{u} + i\vec{v}$ in $\mathbb{C}^2 \Rightarrow Re(\langle x+iy, u+iv\rangle) = 0$

$\langle \vec{x} + i\vec{y}, \vec{u} + i\vec{v}\rangle = \langle \vec{x}, \vec{u}+i\vec{v}\rangle + i\langle \vec{y}, \vec{u}+i\vec{v}\rangle$

$= \langle \vec{x}, \vec{u}\rangle - i\langle \vec{x}, \vec{v}\rangle + i\langle y, \vec{u}\rangle + \langle \vec{y}, \vec{v}\rangle$

$Re(\langle \vec{x}+i\vec{y}, \vec{u}+i\vec{v}\rangle) = \langle \vec{x}, \vec{u}\rangle + \langle \vec{y}, \vec{v}\rangle$

$= \sum_{j=1}^{n} x_j u_j + \sum_{j=1}^{n} y_j v_j = (x,y)\cdot(u,v)$

if $(x,y) \perp (u,v)$ then $(x,y)\cdot(u,v) = 0 = Re(\langle x+iy, u+iv\rangle)$

2\. i $\|cf\| = \sqrt{\langle cf, cf\rangle} = \sqrt{c\langle f, cf\rangle} = \sqrt{c\cdot\overline{c}\,\langle f,f\rangle} = |c|\,\|f\|$

ii $\|f+g\|^2 = \|f\|^2 + \|g\|^2$ iff $f \perp g$.

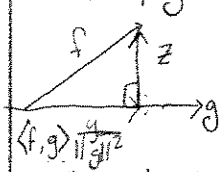
$\|f+g\|^2 = \langle f+g, f+g\rangle = \langle f, f+g\rangle + \langle g, f+g\rangle + \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle$

$= \|f\|^2 + \|g\|^2 + \langle f,g\rangle + \overline{\langle f,g\rangle}$

$= \|f\|^2 + \|g\|^2 + 2\,Re(\langle f,g\rangle)$, $\quad Re(\langle f,g\rangle) = 0$ iff $f \perp g$.

$f - \langle f,g\rangle \frac{g}{\|g\|^2} \cdot g) = 0$

iii. $|\langle f,g\rangle| \leq \|f\|\,\|g\|$



Should check $z \perp g$ also note $=$ iff $z = 0$.

$\|f\| \geq \|\langle f,g\rangle \frac{g}{\|g\|^2}\| = \cancel{\frac{|\langle f,g\rangle|}{\|g\|^2}\|g\|}$

$\|f\|\,\|g\| \geq |\langle f,g\rangle|$

iv. $\|f+g\| \leq \|f\| + \|g\|$

$\|f+g\|^2 = \langle f+g, f+g\rangle = \langle f,f\rangle + \langle g,g\rangle + \langle f,g\rangle + \langle g,f\rangle \leq$

By C.S. Inequality $\leq \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\|$

$= (\|f\| + \|g\|)^2$

2, v. define $d(f,g) = \|f-g\|$

A. $d(f,h) \le d(f,g) + d(g,h)$

R.H.S. $d(f,g) + d(g-h) = \|f-g\| + \|g-h\| \ge$ (by triangle inequality)

$\ge \|f-g+g-h\| = \|f-h\| = d(f,h)$ ✓

B. $d(f,g) \ge 0$, $d(f,g) = 0$ iff $f=g$.

$d(f,g) = \|f-g\| = [\langle f-g, f-g\rangle]^{1/2} \ge 0$ by Ex 1.a.d

$[\langle f-g, f-g\rangle]^{1/2} = 0$ iff $f-g=0 \Rightarrow f=g$.

3. a. $f(t) = t \quad -\pi < t < \pi$

$t \sim \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} t\sin(nt)\,dt \qquad u=t,\ du=dt \qquad dv=\sin(nt)\,dt,\ v=\frac{-\cos(nt)}{n}$

$= \frac{2}{\pi n}\left[-t\cos(nt)\right]_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}\cos(nt)\,dt$

$= \begin{cases} \frac{2}{n} & n \text{ odd} \\ -\frac{2}{n} & n \text{ even} \end{cases}$ $\qquad \boxed{f(t) = t \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nt)}$

b. $\frac{t^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^t \sin(nt)\,dt = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nt) + \underbrace{2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}}_{C_1}$

$\int_0^{\pi}\frac{t^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^{\pi}\cos(nt)\,dt + C_1\int_0^{\pi}dt$

$\Rightarrow \frac{\pi^3}{6} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cancel{[\sin(nt)]_0^{\pi}}^{=0} + C_1(\pi - 0) \Rightarrow C_1 = \frac{\pi^2}{6}$

c. $C_1 = \frac{\pi^2}{6} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}$

d. $\frac{t^4}{24} = \int_0^t\int_0^t 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nt) + \frac{\pi^2}{6}\,dt\,dt$

$= 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^t\left[\sin(nt) + \frac{\pi^2 t}{6}\right]dt \quad C_2$

$\int_0^{\pi}\frac{t^4}{24} = \int_0^{\pi} 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos(nt) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} + \frac{\pi^2 t^2}{12}\,dt$

$\frac{t^5}{120}\Big|_0^{\pi} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\cancel{\sin(nt)}\Big|_0^{\pi} + \frac{\pi^2}{36}t^3\Big|_0^{\pi} + C_2 t\Big|_0^{\pi}$

$\frac{\pi^5}{120} = \frac{\pi^5}{36} + C_2\pi \Rightarrow C_2 = \frac{\pi^4}{120} - \frac{\pi^4}{36} \qquad \boxed{C_2 = \frac{-7\pi^4}{360}}$

e. $C_2 = \frac{+7\pi^4}{360} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}}$

$\frac{t^4}{24} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos(nt) + \frac{-7\pi^4}{360} + \frac{\pi^2 t^2}{12} \qquad t=\pi$

$\frac{\pi^4}{24} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}(-1)^n - \frac{7\pi^4}{360} + \frac{\pi^2\cdot\pi^2}{12} = 2\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n^4} - \frac{7\pi^4}{36} + \frac{\pi^4}{12}$

$\Rightarrow 2\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{12} - \frac{7\pi^4}{360} - \frac{\pi^4}{24} = \frac{\pi^4}{45} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{\pi^4}{90}}$

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \sum_{n\text{ odd}}\frac{1}{n^4} - \sum_{n\text{ even}}^{\infty}\frac{1}{n^4} = \sum_{n\text{ odd}}^{\infty}\frac{1}{n^4} - \sum_{k=1}^{\infty}\frac{1}{(2k)^4} = \sum_{n\text{ odd}}\frac{1}{n^4} - \frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{7\pi^4}{720}$

$\sum_{n\text{ odd}}\frac{1}{n^4} = \frac{7\pi^4}{720} + \frac{1}{16}\left(\frac{\pi^4}{90}\right) = \frac{7\pi^4}{720} + \frac{\pi^4}{1440} \Rightarrow \boxed{\sum_{n\text{ odd}}\frac{1}{n^4} = \frac{\pi^4}{96}}$