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Math 5440
HW 6

① Consider polar coordinates in $\mathbb{R}^2$ i.e. with $r = \sqrt{x^2+y^2}$

$$\theta = \begin{cases} \arctan(y/x) & \text{I, IV} \\ \arccos \frac{x}{\sqrt{x^2+y^2}} & \text{I, II} \\ \text{etc.} \end{cases}$$

a) Show $\ln r = \ln \sqrt{x^2+y^2}$ is harmonic.

Proof: $\ln r = \ln (x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2)$

$$\frac{\partial}{\partial x}\left[\tfrac{1}{2}\ln(x^2+y^2)\right] = \frac{2x}{2(x^2+y^2)} = \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left[\tfrac{1}{2}\ln(x^2+y^2)\right] = \frac{2y}{2(x^2+y^2)} = \frac{y}{x^2+y^2}$$

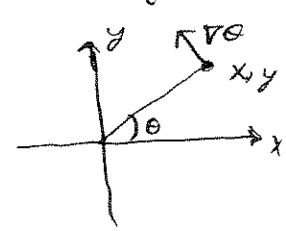
$$\frac{\partial^2}{\partial x^2}\left[\tfrac{1}{2}\ln(x^2+y^2)\right] = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2}{\partial y^2}\left[\tfrac{1}{2}\ln(x^2+y^2)\right] = \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\text{so } \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left[\tfrac{1}{2}\ln(x^2+y^2)\right] = \frac{-x^2+y^2+x^2-y^2}{x^2+y^2} = 0. \quad \square$$

b) Derive the formula below for $\nabla\theta$ without using inverse trig functions:

$$\nabla\theta = \frac{1}{x^2+y^2}\langle -y, x\rangle.$$



Soln: We know that $\nabla\theta$ points in the direction of largest increase of θ. This is in the direction shown in the figure, i.e. perpendicular to the radial direction (the direction in which θ does not increase at all). Since θ increases as we move counterclockwise, $\nabla\theta$ will be in the direction of the counterclockwise rotation of $\langle x, y\rangle$, i.e. $\nabla\theta$ will be in the direction of the unit vector

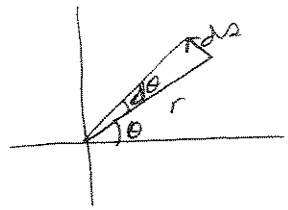
$$\hat{u} = \frac{\langle -y, x\rangle}{\sqrt{x^2+y^2}}.$$

Thus, $\nabla\theta \cdot \hat{u} = \|\nabla\theta\| \|\hat{u}\| \cos(0) = \|\nabla\theta\|$,

since $\nabla\theta$ points in the same direction as $\hat{u}$.

Now, $\nabla\theta = \|\nabla\theta\| \hat{u} = (\nabla\theta \cdot \hat{u})\hat{u}$.

Consider the figure below, where $d\theta$ is an infinitesimal change in θ, ds is an infinitesimal change in arclength, s (approximated by the straight line ds drawn).



$$ds = r\,d\theta \Rightarrow \frac{d\theta}{ds} = \frac{1}{r}$$ But $\hat{u}$ is in the same direction as ds,

so $\|\nabla\theta\| = \nabla\theta \cdot \hat{u} = \frac{d\theta}{ds} = \frac{1}{r}$. Thus, since $\nabla\theta = \|\nabla\theta\|\hat{u}$

$$= \frac{1}{r}\hat{u} = \frac{1}{\sqrt{x^2+y^2}} \frac{\langle -y, x\rangle}{\sqrt{x^2+y^2}} = \frac{1}{x^2+y^2}\langle -y, x\rangle. \quad \square$$

c) Using the formula for $\nabla\theta$ in b, show θ is harmonic.

<u>Proof:</u> $$\Delta\theta = \nabla\cdot(\nabla\theta) = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0. \quad \square$$

d) Show that $\Delta u = u_{xx} + u_{yy} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r^2}u_{\theta\theta}$.

By the chain rule,

$$u_x = u_r r_x + u_\theta \theta_x \qquad u_{xx} = u_r r_{xx} + r_x(u_{rr} r_x + u_{r\theta}\theta_x)$$
$$+ u_\theta \theta_{xx} + \theta_x(u_{\theta r} r_x + u_{\theta\theta}\theta_x)$$

$$u_y = u_r r_y + u_\theta \theta_y \qquad u_{yy} = u_r r_{yy} + r_y(u_{rr} r_y + u_{r\theta}\theta_y)$$

Now, since $r = (x^2+y^2)^{1/2}$ and by part (b),

$$r_x = \frac{x}{(x^2+y^2)^{1/2}} = \frac{x}{r} \qquad r_{xx} = \frac{(x^2+y^2)^{1/2}(1) - x\,\frac{x}{(x^2+y^2)^{1/2}}}{x^2+y^2} = \frac{r - \frac{x^2}{r}}{r^2} = \frac{r^2-x^2}{r^3}$$

$$r_y = \frac{y}{(x^2+y^2)^{1/2}} = \frac{y}{r} \qquad r_{yy} = \frac{(x^2+y^2)^{1/2}(1) - y\,\frac{y}{(x^2+y^2)^{1/2}}}{(x^2+y^2)^2} = \frac{r - \frac{y^2}{r}}{r^2} = \frac{r^2-y^2}{r^3}$$

$$\theta_x = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} \qquad \theta_{xx} = \frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{r^4}$$

$$\theta_y = \frac{x}{x^2+y^2} = \frac{x}{r^2} \qquad \theta_{yy} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{r^4}$$

$$\Rightarrow\ u_{xx} + u_{yy} = u_r \underbrace{(r_{xx}+r_{yy})}_{\frac{r^2-x^2}{r^3}+\frac{r^2-y^2}{r^3} = \frac{2r^2-r^2}{r^3} = \frac{1}{r}} + u_{rr}\underbrace{\left[(r_x)^2+(r_y)^2\right]}_{\frac{x^2}{r^2}+\frac{y^2}{r^2} = 1}$$

$$+2u_{r\theta}\underbrace{(r_x\theta_x + r_y\theta_y)}_{-\frac{xy}{r^3}+\frac{xy}{r^3} = 0} + u_\theta \underbrace{(\theta_{xx}+\theta_{yy})}_{\frac{2xy}{r^4}-\frac{2xy}{r^4} = 0} + u_{\theta\theta}\underbrace{\left[(\theta_x)^2+(\theta_y)^2\right]}_{\frac{y^2}{r^4}+\frac{x^2}{r^4} = \frac{1}{r^2}}$$

$$= \frac{1}{r}u_r + u_{rr} + \frac{1}{r^2}u_{\theta\theta}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r^2}u_{\theta\theta}, \qquad \text{since } \frac{1}{r}\frac{\partial}{\partial r}(r u_r) = \frac{1}{r}(r u_{rr} + u_r) = \frac{1}{r}u_r + u_{rr}.$$

$\square$

e) Use (d) to recheck that $\ln r$ and θ are harmonic functions.

Soln: $\Delta(\ln r) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(\ln r)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\ln r) = \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r\cdot\frac{1}{r}}\right) + 0 = 0+0=$

$\Delta(\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(\theta)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\theta) = 0 + \frac{1}{r^2}\cdot 0 = 0.$ ✓

2) Extended maximum principle in $\mathbb{R}^2$

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with closure $\bar{\Omega} = \Omega \cup \partial\Omega$

Let $w \in C^2(\Omega)$ and continuous except at a finite number of boundary points $p_1, \dots, p_k$.

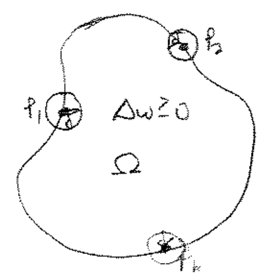
Let w be bounded, $|w(x,y)| \le m \quad \forall (x,y) \in \bar{\Omega}$

Let $\Delta w \ge 0$ in Ω

Then if $w \le \partial\Omega$, $w \le 0$ on $\bar{\Omega}$.

we mean $w \le 0$ on $\partial\Omega$, except at $p_1, p_2, \dots, p_k$ where any definition of w is irrelevant.

Prove this theorem.



Proof: We consider the function

$$v = w + \varepsilon \sum_{j=1}^{k} \ln\left(\frac{\|x - p_j\|}{R}\right),$$

where $\varepsilon > 0$ and R is such that $\Omega \underset{\text{proper}}{\subset} B_R(p_j) \ \forall j$; such an R exists because Ω is bounded. We let v be defined on a subdomain $\Omega_\delta = \Omega \setminus \bigcup_{j=1}^{k} \bar{B}_\delta(p_j)$, where $\delta(\varepsilon) > 0$.

Now, for each $j = 1, \dots, k$ and $\forall x \in \Omega$, $\|x - p_j\| < R$, so $\ln\left(\frac{\|x - p_j\|}{R}\right) < 0$

$\forall x \in \Omega$ and $\forall j$.

Then $v \le w$ on Ω_δ. Additionally we have, by problem 1 and the remarks that follow, that $\ln\left(\frac{\|x - p_j\|}{R}\right) = \ln(\|x - p_j\|) - \ln(R)$ is harmonic. Thus $\Delta v = \Delta w \ge 0$ in Ω_δ.

Now, since $w \le 0$ on $\partial\Omega \setminus \{p_1, \dots, p_k\}$, $v \le 0$ everywhere on $\partial\Omega_\delta$ except possibly on $\bigcup_{j=1}^{k} \partial B_\delta(p_j)$. We will show that, if $\delta = Re^{-\frac{m}{\varepsilon}}$, then in fact $v \le 0$ on all of $\partial\Omega_\delta$.

we have for $x \in \bigcup_{j=1}^{k} \partial \bar{B}_\delta(P_j)$

$$v = w + \varepsilon \sum_{j=1}^{k} \ln\left(\frac{\|x-P_j\|}{R}\right)$$

$$\le M + \varepsilon \sum_{j=1}^{k} \ln\left(\frac{\|x-P_j\|}{R}\right)$$

$$\le M + \varepsilon \ln\left(\frac{\|x-P_j\|}{R}\right) \quad \text{for some } 1 \le j \le k.$$

$$= M + \varepsilon \ln\left(\frac{\delta}{R}\right)$$

$$= M + \varepsilon \ln\left(\frac{R e^{-\frac{M}{\varepsilon}}}{R}\right)$$

$$= M + \varepsilon\left(-\frac{M}{\varepsilon}\right)$$

$$= 0,$$

so $v \le 0$ on $\partial \Omega_\delta$.

Thus, by the maximum principle,

$v \le 0$ on $\overline{\Omega_\delta}$.

Fix $x \in \overline{\Omega_\delta}$. Then, as $\varepsilon \to 0$, $\overline{\Omega_\delta} \to \overline{\Omega}$.

Also, $\lim_{\varepsilon \to 0} v \le \lim_{\varepsilon \to 0} 0 = 0$. Since $v = w + \varepsilon \sum_{j=1}^{k} \ln\left(\frac{\|x-P_j\|}{R}\right)$

$v \xrightarrow[\varepsilon \to 0]{} w$. Thus $w \le 0$ on $\overline{\Omega}$.

3 a) Show that if

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } 0 < x < l$$

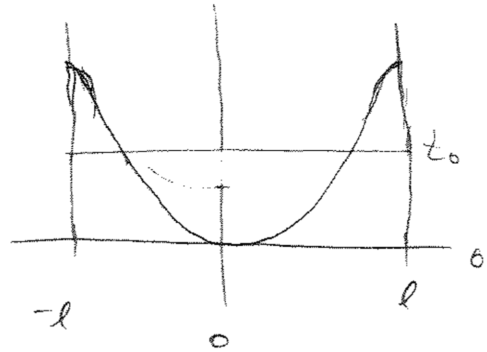
and $\frac{\partial u}{\partial x}\Big|_{0,t} = 0$,

the maximum of u for $0 \le x \le l$ and $0 \le t \le T$ must occur at

$t = 0$ or $x = l$

Soln: If we let $u(x,t) = u(-x,t)$ for $x \in [-l, 0]$, then $u(x,t)$ on $-l \le x \le l$ is an even function such that

$$u_t - k u_{xx} = 0 \quad \text{for } l < x < l.$$



Let $\bar{u}(x,t)$ be the extended function. Then $\bar{u}(x,t)$ solves

$$\bar{u}_t - k\bar{u}_{xx} = 0 \qquad -l < x < l$$

$$\frac{\partial \bar{u}}{\partial x}(0,t) = 0.$$

(This last condition holds because $\bar{u}(x,t)$ is even about $x=0$).
By the maximum principle, (*) see below the maximum value of $\bar{u}$ must be attained on the parabolic boundary described by

$$\left(\partial\Omega_{\bar{u}} \times [0,t_0]\right) \cup \left(\Omega_{\bar{u}} \times \{0\}\right)$$

where $\Omega_{\bar{u}} = (-l, l) \times \{0\}$.

Since $u = \bar{u}\big|_{0 \le x \le l}$, we conclude that the maximum of u must occur on $\left([0,l] \times \{0\}\right) \cup \left(\{l\} \times [0,t_0]\right)$.

(*) Indeed, $\bar{u}$ must be bounded on $-l \le x \le l \times 0 \le t \le t_0$ since $\bar{u}$ is continuous and the set $\{(x,t) \mid -l \le x \le l,\ 0 \le t \le t_0\}$ is compact. Clearly, $\bar{u}$ is C^2 on this set, since u is, i.e., since u is C^2 at 0 from the right, $\bar{u}$ will be C^2 at 0 from the left, so $\bar{u}$ will be C^2 at 0. Thus, the maximum principle applies.

(3b)

3.13.4) Show that a soltn of the nonlinear eqtn

$$\frac{\partial E}{\partial u}(\vec{x}, u(\vec{x},t))\, u_t - \operatorname{div}\left(K(\vec{x}, u(\vec{x},t)) \nabla u\right) = 0$$

solves the max principle provided $\frac{\partial E}{\partial u} > 0$ and $K > 0$

<u>proof</u>: $u(\vec{x},t)$ is given.

define $e(\vec{x}) := \frac{\partial E}{\partial u}(\vec{x}, u(\vec{x},t))$

$\tilde{K}(\vec{x}) := K(\vec{x}, u(\vec{x},t))$

then u also solves the linear PDE

$$L(u) := \boxed{e(\vec{x})\, u_t - \nabla\tilde{K}\cdot\nabla u - K\,\Delta u = 0} \qquad e > 0.$$

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consider

$u^\varepsilon := u + \varepsilon \|x\|^2$

$\frac{\partial u^\varepsilon}{\partial t} = u_t$; $\Delta u^\varepsilon = \Delta u + 2\varepsilon n.$

consider the global maximum value of $u^\varepsilon(\vec{x},t)$ on the compact domain $\bar{\Omega}\times[0,T]$.

If it occurs on $\Omega\times(0,T]$ ~~th~~, say at (x_0,t_0),

then $\nabla u^\varepsilon(x_0,t_0) = 0$

$u^\varepsilon_t(x_0,t_0) \geq 0$ $\quad$ ($=0$ unless $t_0 = T$)

$u^\varepsilon_{\xi\xi}(x_0,t_0) \leq 0$ $\quad$ for all spatial 2^{nd} directional derivatives

$\Rightarrow \Delta u^\varepsilon(x_0,t_0) \leq 0$

$$\Rightarrow L(u^\varepsilon)(x_0,t_0) = \underbrace{e(x)}_{\geq 0}\, u^\varepsilon_t(x_0,t_0) - \underbrace{\nabla\tilde{K}\cdot\nabla u^\varepsilon(x_0,t_0)}_{=0}\ \underbrace{-K\,\Delta u(x_0,t_0)}_{\geq 0}$$

$$\geq 0.$$

but $L(u) = 0$

so $L(u^\varepsilon) = L(u) + L(\varepsilon\|\vec{x}\|^2) = 0 - K\cdot 2\varepsilon n < 0$ $\not\!\!\times$.

Thus the max value of u^ε occurs only on the parabolic boundary $(\partial\Omega\times[0,T]) \cup (\Omega\times\{0\}) := PB$

So for fixed $(\vec{x},t) \in \bar{\Omega}\times[0,T]$ $\quad$ where $\bar{\Omega} \subset B(0,R)$

$$u(x,t) \leq u^\varepsilon(x,t) \leq \max_{PB} u^\varepsilon(x,t) \leq \max_{PB} u(x,t) + \varepsilon R^2$$

$$\lim_{\varepsilon\to 0} \Rightarrow u(x,t) \leq \max_{PB} u(x,t) \quad \blacksquare$$

(4) a) If $\vec{F} = \langle M(x,y), N(x,y)\rangle$ is an $\mathbb{R}^2$ vector field, consider $\mathbb{R}^2$ as the $z=0$ plane in $\mathbb{R}^3$, extend $\vec{F}$ as $\langle M, N, 0\rangle$ in $\mathbb{R}^3$. Let $S = \Omega$ a domain in $\mathbb{R}^2$ and check that Green's Theorem is a special case of Stokes' Theorem.

Soln: Stokes' Theorem states $\iint_{\Omega} (\nabla \times \vec{F}) \cdot \hat{n}\, dA = \oint_{\partial\Omega} \vec{F} \cdot \hat{T}\, ds$.

In this case $\hat{n} = \hat{k}$, and

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = \hat{i}(-N_z) - \hat{j}(-M_z) + \hat{k}(N_x - M_y)$$

Since $\vec{F} \in \mathbb{R}^2$, $N_z = M_z = 0$. Now, $(\nabla \times \vec{F}) \cdot \hat{k} = N_x - M_y$, so Stokes' Theorem becomes

$$\iint_{\Omega} (N_x - M_y)\, dA = \oint_{\partial\Omega} \vec{F} \cdot \hat{T}\, ds,$$

which is Green's Theorem.

b) We need to show Stokes' Theorem for the given graph by invoking Green's Theorem in D, i.e., we need to show that

$$\int_{\partial S} \vec{F} \cdot \hat{T}\, ds = \iint_{S} (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

We begin on the left. We can parameterize the curve ∂S by

$\vec{r}(t) = \langle x(t), y(t), f(x(t), y(t))\rangle$.

Also, $\hat{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{\langle x'(t), y'(t), f_x x'(t) + f_y y'(t)\rangle}{\|\vec{r}\,'(t)\|}$

And, on ∂S, $\vec{F} = \langle M(x(t), y(t), f(x(t), y(t))), N, P\rangle$,

where N and P are also evaluated at $\langle x(t), y(t), f(x(t), y(t))\rangle$.

Now,

$$\oint_{\partial S} \vec{F}\cdot\hat{T}\, ds = \int_t \langle M(x(t),y(t),f(x(t),y(t)),N,P\rangle \cdot \frac{\langle x',y',f_x x'+f_y y'\rangle}{\|\vec{r}\,'(t)\|}\|\vec{r}\,'(t)\|dt$$

since $ds = \|\vec{r}\,'(t)\|dt$.

The above equality is equal to

$$\int_t M(x(t),y(t),f(x(t),y(t)))\,x' + Ny' + P(f_x x' + f_y y')\,dt.$$

Since $x'dt = dx$, $y'dt = dy$, ~~this~~ becomes

$$\int_{\partial D} \left[M(x,y,f(x,y))\,dx + N\,dy + Pf_x\,dx + Pf_y\,dy\right]$$

$$= \int_{\partial D} \left\{\left[M(x,y,f(x,y)) + Pf_x\right]dx + (N+Pf_y)\,dy\right\}$$

By Green's theorem

$$= \iint_D (N+Pf_y)_x - \left[M(x,y,f(x,y)) + Pf_x\right]_y \,dA$$

$$= \iint_D \left[N_x + N_z f_x + \cancel{Pf_{yx}} + P_x f_y - \left(M_y + M_z f_y + \cancel{Pf_{xy}} + P_y f_x\right)\right]dA$$

$$= \iint_D \left[-f_x(P_y - N_z) - f_y(M_z - P_x) + (N_x - M_y)\right] dA$$

$$= \iint_S \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle \cdot \underbrace{\langle -f_x, -f_y, 1\rangle \frac{1}{\sqrt{f_x^2+f_y^2+1}}}_{\hat{n}}\, dS$$

Now, $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \hat{i}(P_y - N_z) + \hat{j}(M_z - P_x) + \hat{k}(N_x - M_y)$,

so the above integral becomes

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS. \quad \blacksquare$$