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Math 5440
Homework #5

2.8.1a. $L[u] = \partial_t^2 + \partial_x\partial_t + \partial_x^2 \qquad B^2 - 4AC = 1-4 = -3$

$\boxed{\text{Elliptic no characteristics.}}$

$\partial_t^2 + \partial_t\partial_x + \partial_x^2 \qquad \partial_\xi = \partial_t + \frac{1}{2}\partial_x \qquad \partial_\eta = \frac{\sqrt{3}}{2}\partial_x$

$= \left(\partial_t^2 + \partial_t\partial_x + \frac{\partial_x^2}{4}\right) + \partial_x^2 - \frac{\partial_x^2}{4} \qquad \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} \qquad \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \eta}$

$= \left(\partial_t + \frac{1}{2}\partial_x\right)^2 + \frac{3}{4}\partial_x^2 \qquad x_\xi = \frac{1}{2} \quad t_\xi = 1 \qquad x_\eta = \frac{\sqrt{3}}{2} \quad t_\eta = 0$

$\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \Rightarrow \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{-2}{\sqrt{3}}\begin{bmatrix} 0 & -\sqrt{3}/2 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix} \qquad \boxed{\xi = t \quad \eta = \frac{2x-t}{\sqrt{3}}}$

$[\partial_x \ \partial_t]\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} \partial_x \\ \partial_t \end{bmatrix} \quad \left|\begin{bmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{bmatrix}\right| = 0 = (1-\lambda)^2 - \frac{1}{4} \Rightarrow 1-\lambda = \pm\frac{1}{2}, \ -\lambda = \frac{1}{2}, \frac{3}{2}$

$\lambda = \frac{1}{2} \quad \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda = \frac{3}{2} \quad \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad O = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = O^T$

$O^TAO = \begin{bmatrix} 1/2 & 0 \\ 0 & 3/2 \end{bmatrix} \qquad \vec{\xi} = O^T\vec{x} \qquad \xi = \frac{-x+t}{\sqrt{2}} \quad \eta = \frac{x+t}{\sqrt{2}} \qquad L[u] = \frac{1}{2}\bar{u}_{\xi\xi} + \frac{3}{2}u_{\eta\eta}.$

b. $L[u] = \partial_{tt} + 4\partial_x\partial_t + 4\partial_{xx} \qquad \boxed{\text{Parabolic Characterics: } C_1 = -1 + \frac{x}{2}}$

$= (\partial_t + 2\partial_x)^2 \qquad \partial_\xi = \partial_t + 2\partial_x \qquad \partial_\eta = \partial_t$

$x_\xi = 2 \quad t_\xi = 1 \quad x_\eta = 0 \quad t_\eta = 1 \qquad \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} \qquad \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \eta}$

$\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \Rightarrow \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix} \Rightarrow \xi = \frac{x}{2} \quad \eta = -\frac{x}{2} + t \quad \eta(0,1) = 1 \quad \eta = -\frac{x}{2} + t = 1 \Rightarrow t$

$\frac{\partial^2 u}{\partial \xi^2} = 0 \Rightarrow u(\xi,\eta) = p(\eta) + \xi q(\eta) \Rightarrow \boxed{u(x,t) = p\left(-\frac{x}{2}+t\right) + \frac{x}{2}q\left(-\frac{x}{2}+t\right)}$

$[\partial_x \ \partial_t]\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \partial_x \\ \partial_t \end{bmatrix} \quad \left|\begin{bmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}\right| = (4-\lambda)(1-\lambda) - 4 = 0 \Rightarrow \lambda = 0, 5$

$\lambda = 0 \quad \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \lambda = 5 \quad \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad O = \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = O^T \qquad O^TAO = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

$\vec{\xi} = O^T\vec{x} \Rightarrow \xi = \frac{-x+2t}{\sqrt{5}} \quad \eta = \frac{2x+t}{\sqrt{5}} \qquad \partial_\eta^2 u = 0 \Rightarrow \bar{u} = \eta\tilde{p}(\xi) + \tilde{q}(\xi)$

$\Rightarrow u(x,t) = \frac{2x+t}{\sqrt{5}}\tilde{p}\left(\frac{-x+2t}{\sqrt{5}}\right) + \tilde{q}\left(\frac{-x+2t}{\sqrt{5}}\right) \qquad 2x+t = \frac{3}{2}x + \left(-\frac{x}{2}+t\right) \qquad \frac{-x+2t}{2} = -\frac{x}{2}+t$

$u(x,t) = \frac{x}{2}\underbrace{3p^*\left(-\frac{x}{2}+t\right)}_{q\left(-\frac{x}{2}+t\right)} + \underbrace{\left(-\frac{x}{2}+t\right)p^*\left(-\frac{x}{2}+t\right) + q^*\left(-\frac{x}{2}+t\right)}_{p\left(-\frac{x}{2}+t\right)} = \frac{x}{2}q\left(-\frac{x}{2}+t\right) + p\left(-\frac{x}{2}+t\right)$

c. $L[u] = \partial_t^2 - 4\partial_t\partial_x + \partial_x^2 \qquad \partial_\xi = \partial_t - (2+\sqrt{3})\partial_x \qquad \partial_\eta = \partial_t - (2-\sqrt{3})\partial_x$

$(\partial_t^2 - 4\partial_t\partial_x + 4\partial_x^2) - 3\partial_x^2)u = 0 \qquad \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} \qquad \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \eta}$

$(\partial_t - 2\partial_x)^2 - 3\partial_x^2 \qquad x_\xi = -2-\sqrt{3} \quad t_\xi = 1 \qquad x_\eta = -2+\sqrt{3} \quad t_\eta = 1$

$(\partial_t - (2+\sqrt{3})\partial_x)(\partial_t - (2-\sqrt{3})\partial_x)u = 0 \qquad \boxed{\text{Hyperbolic } C_1 = 2\sqrt{3} - \frac{x}{2+\sqrt{3}} \quad C_2 = 2\sqrt{3} - \frac{x}{2-\sqrt{3}}}$

$\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} -2-\sqrt{3} & -2+\sqrt{3} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \Rightarrow \xi = \frac{1}{2\sqrt{3}}[-x - (2-\sqrt{3})t], \quad \eta = \frac{1}{2\sqrt{3}}[x + (2+\sqrt{3})t]$

$x + (2+\sqrt{3})t = 4\sqrt{3} + 6 \Rightarrow 2\sqrt{3} - \frac{x}{2+\sqrt{3}} \qquad -x + (-2+\sqrt{3})t = -4\sqrt{3} + 6 \Rightarrow t = 2\sqrt{3} + \frac{x}{-2+\sqrt{3}}$

$u = p(\xi) + q(\eta) \Rightarrow u(x,t) = p\left(\frac{1}{2\sqrt{3}}(-x - (2+\sqrt{3})t)\right) + q\left(\frac{1}{2\sqrt{3}}(x + (2+\sqrt{3})t)\right)$

$u(x,t) = \tilde{p}(x + (2-\sqrt{3})t) + \tilde{q}(x + (2+\sqrt{3})t)$

Continued on back $\longrightarrow$

$$[\partial_x \ \partial_t]\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}\begin{bmatrix}\partial_x\\ \partial_t\end{bmatrix} \qquad \begin{vmatrix}1-\lambda & -2\\ -2 & 1-\lambda\end{vmatrix} = (1-\lambda)^2-4=0 \quad \lambda = 1\pm 2, \quad \lambda = -1, 4$$

$$\lambda=3 \ \begin{bmatrix}-2 & -2\\ -2 & -2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow v_1=\begin{bmatrix}-1\\ 1\end{bmatrix} \qquad \lambda=-1 \ \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow v_2=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$O=O^T=\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} \qquad O^TAO=\begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix} \Rightarrow 3\partial_{\xi\xi}-\partial_{\eta\eta}=0$$

$$\Rightarrow \partial_{\eta\eta}-(\sqrt{3})^2\partial_{\xi\xi}=0 \Rightarrow \bar{u}(\xi,\eta)=p(\xi-\sqrt{3}\eta)+q(\xi+\sqrt{3}\eta)$$

$$\vec{\xi}=O^T\vec{x} \Rightarrow \xi=\frac{-1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}t \qquad \eta=\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}t$$

$$\bar{u}(x,t)=p\left(\frac{-1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}t-\sqrt{\frac{3}{2}}x-\sqrt{\frac{3}{2}}t\right)+q\left(\frac{-1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}t+\sqrt{\frac{3}{2}}x+\sqrt{\frac{3}{2}}t\right),$$

$$=\bar{p}\left((1+\sqrt{3})x-(1-\sqrt{3})t\right)+\bar{q}\left((1-\sqrt{3})x-(1+\sqrt{3})t\right)$$

We must now show that $u(x,t)=\bar{u}(x,t)$ if that is the case then the coefficients of $\bar{p}(x,t)=\tilde{p}(\alpha x,\alpha t)$ and $\bar{q}(x,t)=\tilde{q}(\beta x,\beta t)$

$$(1+\sqrt{3})x-(1-\sqrt{3})t=\alpha x+\alpha(2-\sqrt{3})t$$

$$\alpha=(1+\sqrt{3}) \qquad \alpha(2-\sqrt{3})=(1+\sqrt{3})(2-\sqrt{3})=2-\sqrt{3}+2\sqrt{3}-3=-(1-\sqrt{3}) \checkmark$$

$$(1-\sqrt{3})x-(1+\sqrt{3})t=\beta x+\beta(2+\sqrt{3})t$$

$$\beta=1-\sqrt{3} \qquad \beta(2+\sqrt{3})=(1-\sqrt{3})(2+\sqrt{3})=2+\sqrt{3}-2\sqrt{3}-3=-(1+\sqrt{3}) \checkmark$$

3.11.1

$$*\begin{cases}\nabla^2 u = -F(x,y,z) & \text{on } D \\ u = f & \text{on } \partial D\end{cases}$$

Let u & v solve $*$ set $w = u - v$ then w solves the following

$$\begin{cases}\nabla^2 w = 0 & \text{on } D \\ w = 0 & \text{on } \partial D\end{cases}$$

$$\int_D \underbrace{\operatorname{div}(w \nabla w)}_{\parallel} \, dA = \int_D \nabla w \cdot \nabla w + \cancel{w \nabla^2 w} \, dA$$

$\int_{\partial D} w \nabla w \cdot \hat{n} \, ds = 0$ because $w = 0$ on ∂D.

So $\int_D (|\nabla w|)^2 \, dA = 0$ $\quad (|\nabla w|)^2 \geq 0 \ \forall w$ so $(|\nabla w|)^2 = 0$

$w_x^2 + w_y^2 + w_z^2 = 0 \Rightarrow w_x + w_y + w_z = 0 \Rightarrow w = \text{const} \Rightarrow w = 0$ because $w = 0$ on ∂D.

So there can be at most 1 solution to $*$.

3.11.2.

$$*\begin{cases}\nabla^2 u = F(x,y) & \text{on } D \\ u = f & \text{on } C_1 \\ \frac{\partial u}{\partial n} + \alpha u = 0 & \text{on } C_2\end{cases}$$

Let u & v solve $*$ set $w = u - v$

then w solves the following:

$$\begin{cases}\nabla^2 w = 0 & \text{on } D \\ w = 0 & \text{on } C_1 \\ \frac{\partial w}{\partial n} + \alpha w = 0 & \text{on } C_2\end{cases}$$

$$\int_D \operatorname{div}(w \nabla w) \, dA = \int_D \nabla w \cdot \nabla w + \cancel{w \nabla^2 w} \, dA = \int_D |\nabla w|^2 \, dA$$

$$\parallel$$

$$\cancel{\int_{C_1} w \nabla w \cdot \hat{n} \, ds} + \int_{C_2} w \nabla w \cdot \hat{n} \, ds \ \text{(A)}$$

$|\nabla w|^2 \geq 0 \ \forall w$

(A): $\int_{C_2} w \nabla w \cdot \hat{n} \, ds$ $\quad \nabla w \cdot \hat{n} + \alpha w = 0 \Rightarrow \nabla w \cdot \hat{n} = -\alpha w$

$= \int_{C_2} w(-\alpha w) \, ds = + \int_{C_2} -\alpha w^2 \, ds$ $\quad -\alpha w^2 \leq 0 \ \forall w$

So $|\nabla w|^2 = -\alpha w^2 = 0 \Rightarrow \boxed{w = 0}$

3.11.3.

$$X\begin{cases}\frac{\partial}{\partial x}(e^x u_x) + \frac{\partial}{\partial y}(e^y u_y) = 0 & \text{for } x^2+y^2<1 \\ u = x^2 & \text{for } x^2+y^2=1\end{cases}$$

let u & v solve * set $w = u - v$ then w solves:

$$\begin{cases}\nabla \cdot \langle e^x w_x + e^y w_y \rangle = 0 & \text{on } D \\ w(x,y) = 0 & \text{on } \partial D\end{cases}$$

let $F = \langle e^x w_x + e^y w_y \rangle$

$$\int_D \operatorname{div}(w\vec{F})\,dA = \int_D (\nabla w \cdot \vec{F} + \cancel{w(\nabla \cdot \vec{F})}^{\,0})\,dA$$

$$\|$$

$\int_{\partial D} w\vec{F} \cdot \hat{n}\,ds = 0$ because $w = 0$ on ∂D.

$$\Rightarrow \int_D \nabla w \cdot \vec{F}\,dA = 0$$

$$= \int_D (w_x, w_y) \cdot (e^x w_x, e^y w_y)\,dA$$

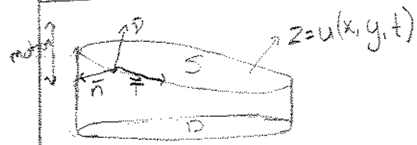
$$= \int_D \underbrace{e^x (w_x)^2 + e^y (w_y)^2}_{\geq 0\ \forall w}\,dA = 0$$

$$\Rightarrow e^x (w_x)^2 + e^y (w_y)^2 = 0$$

$$w_x = w_y = 0 \Rightarrow w = \text{const}$$

$w = 0$ because $w = 0$ on boundry.

Class Exercise 1



$\sigma = \sigma_0 \left[\frac{\text{force}}{\text{length}}\right]$ $\qquad \hat{n} = \vec{T} \times \vec{\nu}$

$\rho = \rho_0 \left[\frac{\text{mass}}{\text{area}}\right]$

Tension force: $f = \oint_{\partial S} \sigma_0 \hat{n}\, ds$

$f^z = \vec{e}_3 \cdot \int_{\partial S} \sigma_0 \hat{n}\, ds$

Stoke's Thm

$\int_{\partial S} \vec{F} \cdot \vec{T}\, ds = \int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{\nu}\, ds$

Body force $\int_D \rho_0 F^z\, dA$

$\sum F = m \frac{dp}{dt} = \frac{d}{dt} \int_D \rho_0 u_t\, dA = \int_D \rho_0 u_{tt}\, dA$

$f^z = \int_{\partial S} \sigma_0 \hat{n} \cdot \vec{e}_3\, ds = \int_{\partial S} \sigma_0 (\vec{T} \times \vec{\nu}) \cdot \vec{e}_3\, ds \qquad (A \times B) \cdot C = (C \times A) \cdot B$

$= \int_{\partial S} \sigma_0 (\vec{e}_3 \times \vec{\nu}) \cdot \vec{T}\, ds \qquad \vec{\nu} = \frac{\langle -u_x, u_y, 1 \rangle}{\sqrt{1 + \|\nabla u\|^2}}$

$= \sigma_0 \int_S \nabla \times (\vec{e}_3 \times \vec{\nu}) \cdot \vec{\nu}\, ds \qquad \vec{e}_3 \times \vec{\nu} = \frac{1}{\sqrt{1+\|\nabla u\|^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -u_x & u_y & 1 \end{vmatrix} = [\hat{i}(-u_y) + \hat{j}(u_x)] \frac{1}{\sqrt{1+\|\nabla u\|^2}}$

$\nabla \times (\vec{e}_3 \times \vec{\nu}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -u_y & u_x & 0 \end{vmatrix}$ because we are going to linearize in the end, we can ignore the $\frac{1}{\sqrt{1+\|\nabla u\|^2}}$ because all contributions from it will be $\mathcal{O}(\varepsilon^2)$ so they will vanish.

$= \cancel{\hat{i}(u_{xz})}^{0} + \hat{j}\cancel{(u_{yz})}^{0} + \hat{k}(u_{xx} + u_{yy})$

$= \sigma_0 \int_S \hat{k}(\Delta u) \cdot \frac{\langle -u_x, u_y, 1 \rangle}{\sqrt{1+\|\nabla u\|^2}}\, ds$

$= \int_S \sigma_0 \frac{\Delta u}{\sqrt{1+\|\nabla u\|^2}}\, dS \qquad ds = \sqrt{1 + \|\nabla u\|^2}\, dA$

$= \int_D \sigma_0 \Delta u\, dA$

$\int_D \sigma_0 \Delta u\, dA + \int_D \rho_0 F^z\, dA = \int_D \rho_0 u_{tt}\, dA$

$\Rightarrow \int_D [\sigma_0 \Delta u + \rho_0 F^z - \rho_0 u_{tt}]\, dA = 0$

$\Rightarrow \sigma_0 \Delta u + \rho_0 F^z - \rho_0 u_{tt} = 0$

$\Rightarrow u_{tt} - \frac{\sigma_0}{\rho_0} \Delta u = F^z \Rightarrow u_{tt} - c^2 \Delta u = F^z \qquad c = \sqrt{\sigma_0 / \rho_0}$