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Math 5440
HW 4

Section 2.6: #1

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0<x<1,\ t>0 \\ u(x,0) = 0 & 0 \le x \le 1 \\ u_t(x,0) = 0 & 0 \le x < 1 \\ u(0,t) = \sin^2 t \\ u(1,t) = 0 \end{cases}$$

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$\sin^2 t$

$2\sin t \cos t = \sin 2t$

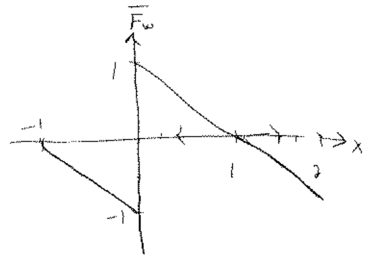
find $u(\frac{1}{2}, \frac{3}{2})$

Soln: Let $v(x,t) = (1-x)\sin^2(t)$. Then $v(0,t) = \sin^2(t)$
$v(1,t) = 0$

Let $w = u - v$. Then w solves:

$$\begin{cases} w_{tt} - w_{xx} = u_{tt} - u_{xx} - (v_{tt} - v_{xx}) = -2(1-x)\cos(2t) := F_w(x,t). \\ w(x,0) = u(x,0) - v(x,0) = 0 - 0 = 0 \\ w_t(x,0) = u_t(x,0) - v_t(x,0) = 0 - 2(1-x)\sin(0)\cos(0) = 0 \\ w(0,t) = u(0,t) - v(0,t) = \sin^2 t - \sin^2 t = 0 \\ w(1,t) = u(1,t) - v(1,t) = 0 - 0 = 0. \end{cases}$$

Since the boundary conditions are Dirichlet, we need to extend F_w in an odd fashion about 0 and 1. We can focus on the function $(1-x)$.



The odd extension is:

$$\begin{cases} 1-x & 0 \le x \le 1 \\ -1-x & -1 \le x \le 0 \\ 1-x & 1 \le x \le 2 \end{cases}$$

For $-1 \le x \le 0$,

$$-f(-x) = -[1-(-x)] = -1-x$$

For $1 \le x \le 2$

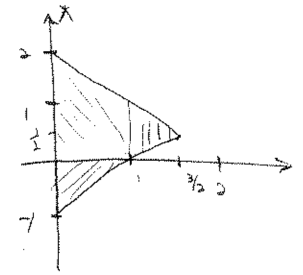
$$-f(2-x) = -[1-(2-x)] = -(-1+x) = (1-x)$$

Therefore, by formula (5.2),

$$w\left(\tfrac{1}{2}, \tfrac{3}{2}\right) = \frac{1}{2}\int_{\bar t=0}^{\frac{3}{2}} \int_{\bar x = \frac{1}{2} - (\frac{3}{2} - \bar t)}^{\frac{1}{2} + (\frac{3}{2} - \bar t)} F_w(\bar x, \bar t)\, d\bar x\, d\bar t$$

$$= \frac{1}{2}\int_{\bar t=0}^{\frac{3}{2}} \int_{\bar x=-1+\bar t}^{2-\bar t} F_w(\bar x, \bar t)\, d\bar x\, d\bar t$$

The region of integration is illustrated below:



Thus
$$w\left(\tfrac{1}{2}, \tfrac{3}{2}\right) = \frac{1}{2}\int_{\bar t=0}^{1} \int_{\bar x=-1+\bar t}^{0} -2(-1-\bar x)\cos(2\bar t)\, d\bar x\, d\bar t + \frac{1}{2}\int_{\bar t=0}^{1} \int_{\bar x=0}^{2-\bar t} -2(1-\bar x)\cos(2\bar t)\, d\bar x\, d\bar t$$

$$+ \frac{1}{2}\int_{\bar t=1}^{\frac{3}{2}} \int_{\bar x=-1+\bar t}^{2-\bar t} -2(1-\bar x)\cos(2\bar t)\, d\bar x\, d\bar t$$

from Maple (see last page)

$$= -\tfrac{1}{2}\cos(2) + \tfrac{1}{4} + \tfrac{1}{4}\cos(3)$$

And
$$u\left(\tfrac{1}{2}, \tfrac{3}{2}\right) = w\left(\tfrac{1}{2}, \tfrac{3}{2}\right) + v\left(\tfrac{1}{2}, \tfrac{3}{2}\right) = -\tfrac{1}{2}\cos(2) + \tfrac{1}{4} + \tfrac{1}{4}\cos(3) + \left(1-\tfrac{1}{2}\right)\sin^2\left(\tfrac{3}{2}\right)$$

$$= -\tfrac{1}{2}\cos(2) + \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(3)\right) + \tfrac{1}{2}\sin^2\left(\tfrac{3}{2}\right)$$

$$= -\tfrac{1}{2}\cos(2) + \tfrac{1}{2}\cos^2\left(\tfrac{3}{2}\right) + \tfrac{1}{2}\sin^2\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\left[\cos^2\left(\tfrac{3}{2}\right) + \sin^2\left(\tfrac{3}{2}\right)\right] - \tfrac{1}{2}\cos(2)$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\cos(2) = \boxed{\sin^2(1)}$$

(3) 2/2 Show that if $f(0)=f(\ell)=g(0)=g(\ell)=0$, the function $v=f(x)+tg(x)$ satisfies all the initial and boundary conditions of (5.3). Show that if the solution of the problem for $w=u-v$ is obtained by means of (5.2), the function $u=w+v$ so derived agrees with the solution by (5.2) of the original problem.

$$(5.3)\quad u_{tt}-c^2u_{xx}=F(x,t)\quad 0<x<\ell,\ t>0$$
$$u(x,0)=f(x)\quad 0<x<\ell$$
$$u_t(x,0)=g(x)$$
$$u(0,t)=0$$
$$u(\ell,t)=0.$$

Soln: Let $v(x,t)=f(x)+tg(x)$. Then

$v(x,0)=f(x)+0\cdot g(x)=f(x)$,
$v_t(x,0)=0+g(x)=g(x)$,
$v(0,t)=f(0)+tg(0)=0+0=0$,
$v(\ell,t)=f(\ell)+tg(\ell)=0+0=0$.

Let $w=u-v$. Then

$$w_{tt}-c^2w_{xx}=u_{tt}-c^2u_{xx}-(v_{tt}-c^2v_{xx})$$
$$=F(x,t)-\{-c^2[f''(x)+tg''(x)]\}$$
$$=F(x,t)+c^2[f''(x)+tg''(x)]$$

$w(x,0)=u(x,0)-v(x,0)=f(x)-f(x)=0$
$w_t(x,0)=u_t(x,0)-v_t(x,0)=g(x)-g(x)=0$
$w(0,t)=u(0,t)-v(0,t)=0-0=0$
$w(\ell,t)=u(\ell,t)-v(\ell,t)=0-0=0.$

Let F, f, and g now denote the odd extensions of F, f, and g about 0 and ℓ, respectively.

By (5.2), $w(x,t)=\frac{1}{2c}\int_{\bar t=0}^{t}\int_{\bar x=x-c(t-\bar t)}^{x+c(t-\bar t)}\{F(\bar x,\bar t)+c^2[f''(\bar x)+\bar t g''(\bar x)]\}\,d\bar x\,d\bar t$

$$=\frac{1}{2c}\left\{\int_{\bar t=0}^{t}\int_{\bar x=x-c(t-\bar t)}^{x+c(t-\bar t)}F(\bar x,\bar t)\,d\bar x\,d\bar t+c^2\int_{\bar t=0}^{t}\int_{\bar x=x-c(t-\bar t)}^{x+c(t-\bar t)}[f''(\bar x)+\bar t g''(\bar x)]\,d\bar x\,d\bar t\right\}$$

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$$=\frac{1}{2c}\left\{\int_{\bar t=0}^{t}\int_{\bar x=x-c(t-\bar t)}^{x+c(t-\bar t)}F(\bar x,\bar t)\,d\bar x\,d\bar t+c^2\int_{\bar t=0}^{t}[f'(\bar x)+\bar t g'(\bar x)]_{\bar x=x-c(t-\bar t)}^{x+c(t-\bar t)}\,d\bar t\right\}$$

$$=\frac{1}{2c}\Big(A(x,t)+c^2\int_{\bar t=0}^{t}\{f'[x+c(t-\bar t)]+\bar t g'[x+c(t-\bar t)]-f'[x-c(t-\bar t)]-\bar t g'[x-c(t-\bar t)]\}$$

$$=\frac{1}{2c}\Big(A(x,t)+c^2\int_{\bar t=0}^{t}\{f'[x+c(t-\bar t)]-f'[x-c(t-\bar t)]\}\,d\bar t$$
$$+c^2\int_{\bar t=0}^{t}\{\bar t g'[x+c(t-\bar t)]-\bar t g'[x-c(t-\bar t)]\}\,d\bar t\Big)$$

$u_0=x+c(t-\bar t)$, $du_0=-c\,d\bar t$; $u_1=x-c(t-\bar t)$, $du_1=c\,d\bar t$

$u_2=\bar t$, $du_2=d\bar t$; $v_2=-\frac{1}{c}g[x+c(t-\bar t)]$, $dv_2=g'[x+c(t-\bar t)]\,d\bar t$

$u_3=\bar t$, $du_3=d\bar t$; $v_3=\frac{1}{c}g[x-c(t-\bar t)]$, $dv_3=g'[x-c(t-\bar t)]\,d\bar t$

$$=\frac{1}{2c}\Big(A(x,t)+c^2\Big[\int_{u_0=x+ct}^{x}-\frac{1}{c}f'(u_0)\,du_0-\int_{u_1=x-ct}^{x}\frac{1}{c}f'(u_1)\,du_1\Big]$$
$$+c^2\Big\{-\frac{\bar t}{c}g[x+c(t-\bar t)]\Big|_{\bar t=0}^{t}-\int_{\bar t=0}^{t}-\frac{1}{c}g[x+c(t-\bar t)]\,d\bar t\Big\}$$
$$-c^2\Big\{\frac{\bar t}{c}g[x-c(t-\bar t)]\Big|_{\bar t=0}^{t}-\int_{\bar t=0}^{t}\frac{1}{c}g[x-c(t-\bar t)]\,d\bar t\Big\}\Big)$$

$$=\frac{1}{2c}\Big(A(x,t)-cf(u_0)\Big|_{u_0=x+ct}^{x}-cf(u_1)\Big|_{u_1=x-ct}^{x}$$
$$+c\Big\{-tg(x)+0+\int_{\bar t=0}^{t}g[x+c(t-\bar t)]\,d\bar t\Big\}$$
$$-c\Big\{tg(x)-0-\int_{\bar t=0}^{t}g[x-c(t-\bar t)]\,d\bar t\Big\}\Big)$$

$$=\frac{1}{2c}\Big\{A(x,t)-c[f(x)-f(x+ct)]-c[f(x)-f(x-ct)]-2ctg(x)$$
$$-\int_{u_0=x+ct}^{x}g(u_0)\,du_0+\int_{u_1=x-ct}^{x}g(u_1)\,du_1\Big\}$$

$$= \frac{1}{2c}\Big\{ A(x,t) - 2c f(x) + c\left[f(x+ct) - f(x-ct)\right] - 2ct g(x) + \int_{u=x}^{x+ct} g(u)\,du + \int_{x-ct}^{x} g(u)\,du$$

$$= \frac{1}{2c}\Big\{ A(x,t) - 2c f(x) + c\left[f(x+ct) - f(x-ct)\right] - 2ct g(x) + \int_{x-ct}^{x+ct} g(u)\,du \Big\}$$

Thus, since $u = w + v$, we have

$$u = \frac{1}{2c}\Big\{ A(x,t) - 2c f(x) + c\left[f(x+ct) + f(x-ct)\right] - 2ct g(x) + \int_{x-ct}^{x+ct} g(u)\,du \Big\}$$

$$+ f(x) + t g(x).$$

$$= \frac{1}{2c} A(x,t) - \cancel{f(x)} + \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] - \cancel{t g(x)} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du + \cancel{f(x)} + \cancel{t g(x)}$$

$$= \frac{1}{2c}\int_{\bar{t}=0}^{t}\int_{\bar{x}=x-c(t-\bar{t})}^{x+c(t-\bar{t})} F(\bar{x},\bar{t})\,d\bar{x}\,d\bar{t} + \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du.$$

$\square$

✓

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restart;

$$w := \frac{1}{2}\cdot\left(\int_0^1\int_{-1+t}^{0} 2\cdot(1+x)\cdot\cos(2\cdot t)\,dx\,dt + \int_0^1\int_0^{2-t} -2\cdot(1-x)\cdot\cos(2\cdot t)\,dx\,dt + \int_1^{\frac{3}{2}}\int_{-1+t}^{2-t} -2\cdot(1-x)\cdot\cos(2\cdot t)\,dx\,dt\right);$$

$$-\frac{1}{2}\cos(2) + \frac{1}{4} + \frac{1}{4}\cos(3) \qquad (1)$$

④ Find which of the following operators L are linear
2/2

a) $L[u] = \frac{\partial u}{\partial t} + x^2 \frac{\partial^2 u}{\partial x^2}$

b) $L[u] = \frac{\partial u}{\partial t} + u\frac{\partial^2 u}{\partial x^2} + u$

c) $L[u] = \left(\frac{\partial u}{\partial t}\right)^2 + \frac{\partial^2 u}{\partial x^2}$

d) $L[u] = \frac{\partial^2 u}{\partial t^2} - e^{x^2 t}\frac{\partial^2 u}{\partial x^2} + t^2 u$.

Soln: Let u, v be functions and α, β be constants.

a)
$$L[\alpha u+\beta v] = \frac{\partial(\alpha u+\beta v)}{\partial t} + x^2\frac{\partial^2}{\partial x^2}(\alpha u+\beta v)$$
$$= \alpha\frac{\partial u}{\partial t} + \beta\frac{\partial v}{\partial t} + x^2\alpha\frac{\partial^2 u}{\partial x^2} + \beta x^2\frac{\partial^2 v}{\partial x^2}$$
$$= \alpha\frac{\partial u}{\partial t} + \alpha x^2\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial v}{\partial t} + \beta x^2\frac{\partial^2 v}{\partial x^2}$$
$$= \alpha\left(\frac{\partial u}{\partial t} + x^2\frac{\partial^2 u}{\partial x^2}\right) + \beta\left(\frac{\partial v}{\partial t} + x^2\frac{\partial^2 v}{\partial x^2}\right)$$
✓
$$= \alpha L[u] + \beta L[v], \text{ so } L \text{ is linear.}$$

b) Let $u(x,t) = x^3$. Then, for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$,
$$L[\alpha u] = \frac{\partial}{\partial t}(\alpha x^3) + (\alpha x^3)\frac{\partial^2}{\partial x^2}(\alpha x^3) + \alpha x^3$$
$$= 0 + \alpha x^3(6\alpha x) + \alpha x^3 = 6\alpha^2 x^4 + \alpha x^3.$$
$$\alpha L[u] = \alpha\left(\frac{\partial}{\partial t}(x^3) + x^3\frac{\partial^2}{\partial x^2}(x^3) + x^3\right) = \alpha x^3(6x) + \alpha x^3 = 6\alpha x^4 + \alpha x^3$$

If $\alpha = 2$, then $L[2u] = 24x^4 + 2x^3$ but $2L[u] = 12x^4 + 2x^3$. ✓

Since $L[2u] \neq 2L[u]$, L is not linear.

c) Let $u = t^2$.

Then $L[3u] = \left[\frac{\partial}{\partial t}(3t^2)\right]^2 + \frac{\partial^2}{\partial x^2}(3t^2) = (6t)^2 + 0 = 36t^2$

But $3L[u] = 3\left\{\left[\frac{\partial}{\partial t}(t^2)\right]^2 + \frac{\partial^2}{\partial x^2}(t^2)\right\} = 3(2t)^2 = 3(4t^2) = 12t^2$ ✓

Since $L[3u] \neq 3L[u]$, L is not linear.

d)
$$L[\alpha u+\beta v] = \frac{\partial^2}{\partial t^2}(\alpha u+\beta v) - e^{x^2 t}\frac{\partial^2}{\partial x^2}(\alpha u+\beta v) + t^2(\alpha u+\beta v)$$
$$= \alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial^2 v}{\partial t^2} - e^{x^2 t}\left(\alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2 v}{\partial x^2}\right) + \alpha t^2 u + \beta t^2 v$$
✓
$$= \alpha\left(\frac{\partial^2 u}{\partial t^2} - e^{x^2 t}\frac{\partial^2 u}{\partial x^2} + t^2 u\right) + \beta\left(\frac{\partial^2 v}{\partial t^2} - e^{x^2 t}\frac{\partial^2 v}{\partial x^2} + t^2 v\right)$$
$$= \alpha L[u] + \beta L[v] \Rightarrow L \text{ is linear.} \quad \square$$

⑤ Show that the problem
2/2

$$(*)\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0<x<\ell,\ t>0 \\ u(0,t) = f_1(t) \\ u_x(\ell,t) = f_2(t) \\ u(x,0) = f_3(x) \\ u_t(x,0) = f_4(x) \end{cases}$$

has at most one solution.

Proof: Suppose $u(x,t)$ and $v(x,t)$ are both solutions of $(*)$, and let $w(x,t) = u(x,t) - v(x,t)$. Then w solves:

$$(**)\begin{cases} w_{tt} - c^2 w_{xx} = (u_{tt} - c^2 u_{xx}) - (v_{tt} - c^2 v_{xx}) = 0 - 0 = 0 = F_w \\ w(0,t) = u(0,t) - v(0,t) = 0 - 0 = 0. = f_w \\ w_x(\ell,t) = u_x(\ell,t) - v_x(\ell,t) = 0 - 0 = 0. = g_w \\ w(x,0) = u(x,0) - v(x,0) = 0 - 0 = 0 \\ w_t(x,0) = u_t(x,0) - v_t(x,0) = 0 - 0 = 0. \end{cases}$$

✓ By chapter 1, the unique solution to $(**)$ is
$$w(x,t) = \frac{1}{2}\left[\bar{f}_w(x+ct) + \bar{f}_w(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\bar{g}_w(\xi)\,d\xi + \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)}\bar{F}_w(\xi,\tau)\,d\xi\,d\tau.$$

where $\bar{f}_w$, $\bar{g}_w$, and $\bar{F}_w$ the extensions of f_w, g_w, and F_w which are odd about $x=0$ and even about $x=\ell$. Since f_w, g_w, and F_w are all 0, and since $w(x,0)=0$, $w(x,t) \equiv 0$ $\forall t \geq 0$ and $0 \leq x \leq \ell$. Therefore $u(x,t) = v(x,t)$. This is the only possible solution.

since $w(x,t)$ solves $(**)$ uniquely; the only other possibility is no solution. Thus $(*)$ has at most one solution.

Section 2.7

① Find the characteristics through the point $(0,1)$ for

$$u_{tt} - e^{2x} u_{xx}.$$

1/1

Soln: The characteristics are found from $\left(\frac{dt}{dx}\right)^2 = \frac{1}{c^2(x)}$, where $c^2(x) = e^{2x}$.

They are

$$\frac{dt}{dx} = \frac{1}{e^x} = e^{-x} \qquad\qquad \frac{dt}{dx} = -\frac{1}{e^x} = -e^{-x}$$

$$\int dt = \int e^{-x}dx \qquad\qquad \int dt = \int -e^{-x}dx$$

$$t = -e^{-x} + C_1 \qquad\qquad t = e^{-x} + C_2$$

$$\Rightarrow 1 = -e^{-0} + C_1 \qquad\qquad \Rightarrow 1 = e^{-0} + C_2$$

$$1 = -1 + C_1 \qquad\qquad 1 = 1 + C_2$$

$$2 = C_1 \qquad\qquad C_2 = 0$$

$\Rightarrow$ $\boxed{t = 2 - e^{-x} \text{ and } t = e^{-x} \text{ are the characteristics.}}$

③ Find the domain of dependence of the point $(1/4, 3)$ wrt the problem

1/1

$$\begin{cases} u_{tt} - (1+x^2)^2 u_{xx} = F(x,t) & 0<x<1,\ t>0 \\ u(x,0) = 0 & 0 \le x \le 1 \\ u_t(x,0) = 0 & 0 \le x \le 1 \\ u(0,t) = 0 & \\ u_x(1,t) = 0 & \end{cases}$$

Soln: We need to solve the following seperable ODE's:

$$\frac{dt}{dx} = \frac{1}{1+x^2} \quad \text{and} \quad \frac{dt}{dx} = \frac{-1}{1+x^2}$$

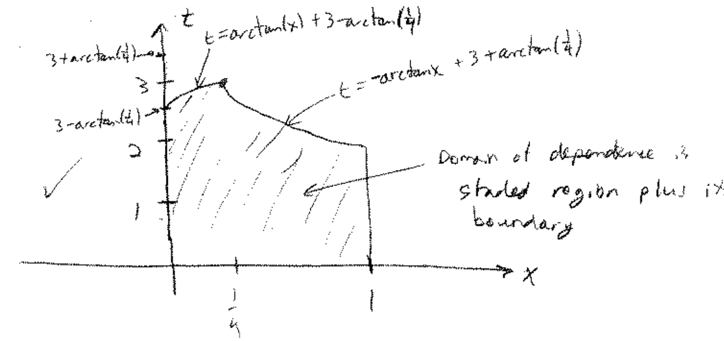
$$\int dt = \int \frac{1}{1+x^2}dx \qquad\qquad \int dt = \int -\frac{1}{1+x^2}dx$$

$$t = \arctan(x) + C_1 \qquad\qquad t = -\arctan(x) + C_2$$

$$3 = \arctan(\tfrac{1}{4}) + C_1 \qquad\qquad 3 = -\arctan(\tfrac{1}{4}) + C_2$$

$$C_1 = 3 - \arctan(\tfrac{1}{4}) \qquad\qquad C_2 = 3 + \arctan(\tfrac{1}{4})$$

$$t = \arctan(x) + 3 - \arctan(\tfrac{1}{4}) \qquad\qquad t = -\arctan(x) + 3 + \arctan(\tfrac{1}{4}).$$



Domain of Dependence $= \{(x,t) \mid 0 \le x \le \frac{1}{4} \text{ and } 0 \le t \le \arctan x + 3 - \arctan(\frac{1}{4})\}$
$\cup \{(x,t) \mid \frac{1}{4} < x \le 1 \text{ and } 0 \le t \le -\arctan x + 3 + \arctan(\frac{1}{4})\}$

1/1

④ Find the domain of influence of the point $(\frac{1}{2}, 0)$ wrt the problem

$$\begin{aligned} u_{tt} - (1+x^2)^2 u_{xx} &= 0 & 0<x<1,\ t>0 \\ u(x,0) &= f(x) & 0 \le x \le 1 \\ u_t(x,0) &= g(x) & 0 \le x \le 1 \\ u(0,t) = u(1,t) &= 0 & \end{aligned}$$

Soln: As before, we first find the characteristics through the point $(\frac{1}{2}, 0)$ by solving

$$\frac{dt}{dx} = \frac{1}{1+x^2} \quad \text{and} \quad \frac{dt}{dx} = \frac{-1}{1+x^2}$$

$\frac{dt}{dx} = \frac{1}{1+x^2}$

$\int dt = \int \frac{1}{1+x^2}\,dx$

$t = \arctan(x) + C_1$

$0 = \arctan(\frac{1}{2}) + C_1$

$C_1 = -\arctan(\frac{1}{2})$

$t_1(x) := \arctan(x) - \arctan(\frac{1}{2})$

$\frac{dt}{dx} = -\frac{1}{1+x^2}$

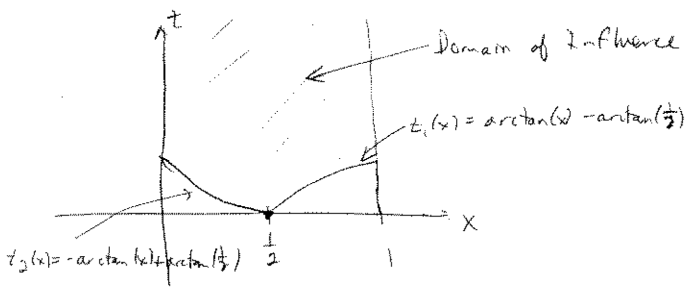
$\int dt = \int -\frac{1}{1+x^2}\,dx$

$t = -\arctan(x) + C_2$

$0 = -\arctan(\frac{1}{2}) + C_2$

$C_2 = \arctan(\frac{1}{2})$

$t_2(x) := -\arctan(x) + \arctan(\frac{1}{2})$



Domain of Influence:

$\{(x,t) \mid 0 \le x \le \frac{1}{2},\ t \ge -\arctan(x) + \arctan(\frac{1}{2})\} \cup \{(x,t) \mid \frac{1}{2} < x \le 1,\ t \ge \arctan(x) - \arctan(\frac{1}{2})\}$

⑥ Show that the domain of dependence of a point $(\bar{x},\bar{t})$ w.r.t the problem

$$(*)\quad \begin{cases} u_{tt} - c^2(x)\,u_{xx} = F(x,t) & 0<x<1,\ t>0 \\ u(x,0) = 0 & 0 \le x \le 1 \\ u_t(x,0) = 0 & 0 \le x \le 1 \\ u(0,t) = 0 & \\ u_x(1,t) + u(1,t) = 0 & \end{cases}$$

is again bounded by the characteristics C_1 and C_2 satisfying

$C_1:\ t = \bar{t} - \int_x^{\bar{x}} \frac{d\xi}{c(\xi)}$

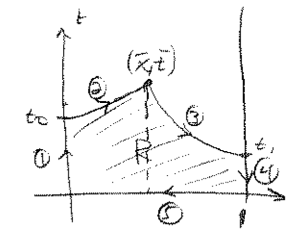
$C_2:\ t = \bar{t} - \int_{\bar{x}}^{x} \frac{d\xi}{c(\xi)}$

Sol'n: We will follow the proof of Theorem 2 outlined in class.

Let u and v be two solutions of $(*)$, and let $w = u - v$.

Then w solves:

$$\begin{cases} w_{tt} - c^2(x)\,w_{xx} = 0 \\ w(x,0) = 0 \\ w_t(x,0) = 0 \\ w(0,t) = 0 \\ w_x(1,t) + w(1,t) = 0 \end{cases}$$



Define the following vector field:

$\langle M, N\rangle = \left\langle -\left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right),\ -w_x w_t \right\rangle$

* We note that, by formula (7.2) in the notes,

$$\left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right)_t - (w_x w_t)_x = 0 \quad (**)$$

Thus $\iint_R (N_x - M_t)\,dA = \iint_R \left[-(w_x w_t)_x + \left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right)_t\right] dA = 0$ by $(**)$

By Green's Theorem, $\iint_R (N_x - M_t)\,dA = \oint_{\partial R} M\,dx + N\,dy$

$= -\oint_{\partial R} M\,dx + N\,dy$

$= -\oint_{\partial R} -\left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right)dx - w_x w_t\,dt$

$= \oint_{\partial R} \underbrace{\left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right)}_{A(x,t)}dx + \underbrace{w_x w_t}_{B(x,t)}\,dt$

$= \int_{①} A(x,t)\,dx + B(x,t)\,dt + \int_{②} A(x,t)\,dx + B(x,t)\,dt + \int_{③} A(x,t)\,dx + B(x,t)\,dt$

$+ \int_{④} A(x,t)\,dx + B(x,t)\,dt + \int_{⑤} A(x,t)\,dx + B(x,t)\,dt.$

Now, $\int_{①} \left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right) dx + w_x w_t \, dt$

$= \int_{t_0}^{t_0} w_x w_t \, dt$, but $w(0,t) = 0$ for $t \geq 0$, so $w_t(0,t) = 0$ for $t \geq 0$.

$= 0$.

$\int_{②} \left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right) dx + w_x w_t \, dt$.

Now, this characteristic is a curve $t = t(x)$, so the integral above becomes:

$$\int_{②} \left[\frac{1}{2}\left(\frac{w_t}{c}\right)^2 + \frac{1}{2} w_x^2 + w_x w_t \frac{dt}{dx}\right] dx$$

Similarly, we obtain $\int_{③} \left[\frac{1}{2}\left(\frac{w_t}{c}\right)^2 + \frac{1}{2} w_x^2 + w_x w_t \frac{dt}{dx}\right] dx$

Now, $\int_{④} \left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right) dx + w_x w_t \, dt$

$= \int_{t_1}^{0} 0 + w_x w_t \, dt = \int_{t_1}^{0} w_x(1,t) w_t(1,t) \, dt$.

Since $w_x(1,t) + w(1,t) = 0$, $w_x(1,t) = -w(1,t)$, so we have

$- \int_{t_1}^{0} w(1,t) w_t(1,t) \, dt$

$= \int_0^{t_1} \frac{1}{2} \frac{d}{dt} [w(1,t)]^2 \, dt$

$= \frac{1}{2} [w(1,t)]^2 \Big|_0^{t_1}$

$= \frac{1}{2} [w(1,t_1)]^2 - \underbrace{\frac{1}{2}[w(1,0)]^2}_{0}$

$= \frac{1}{2} [w(1,t_1)]^2$ ✓

Also, $\int_{①} \left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right) dx + w_x w_t \, dt$

$= \int_1^0 \left\{ \frac{1}{2c^2} \underbrace{[w_t(x,0)]^2}_{0} + \frac{1}{2} \underbrace{[w_x(x,0)]^2}_{0} \right\} dx$ (since $w(x,0) = 0 \;\; \forall x \in [0,1]$) $= 0$

Thus $0 = \oint_{\partial R} \left(\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2\right) dx + w_x w_t \, dt$

$= \int_{②+③} \left[\frac{1}{2}\left(\frac{w_t}{c}\right)^2 + \frac{1}{2} w_x^2 + w_x w_t \frac{dt}{dx}\right] dx + \frac{1}{2}[w(1,t_1)]^2$ (**)

As stated on page 38, uniqueness arguments lead us to consider the case where the integrand is nonnegative. We can complete the square in the integrand as follows:

$\frac{1}{2c^2} w_t^2 + \frac{1}{2} w_x^2 + w_x w_t \frac{dt}{dx}$

$= \frac{1}{2c^2} \left(w_t^2 + c^2 w_x^2 + 2c^2 w_x w_t \frac{dt}{dx} \right)$

$= \frac{1}{2c^2} \left[w_t^2 + 2c^2 w_x w_t \frac{dx}{dt} + c^4 w_x^2 \left(\frac{dt}{dx}\right)^2 \right] + \frac{1}{2} w_x^2 - \frac{1}{2} c^2 w_x^2 \left(\frac{dt}{dx}\right)^2$

$* \; \left(c^2 w_x \cdot \frac{dx}{dt}\right)^2 = c^4 w_x^2 \left(\frac{dt}{dx}\right)^2$

$= \frac{1}{2c^2} \left(w_t + c^2 w_x \frac{dx}{dt} \right)^2 + \frac{1}{2} w_x^2 \left[1 - c^2 \left(\frac{dt}{dx}\right)^2 \right]$. (***)

This last expression is nonnegative if $1 - c^2\left(\frac{dt}{dx}\right)^2$ is nonnegative,

i.e. $1 - c^2\left(\frac{dt}{dx}\right)^2 \geq 0 \iff c^2 \left(\frac{dt}{dx}\right)^2 \leq 1$

$\iff \left(\frac{dt}{dx}\right)^2 \leq \frac{1}{[c(x)]^2}$.

In order to make the characteristics as steep as possible (and, in turn, make the domain of dependence as small as possible), we choose $\left(\frac{dt}{dx}\right)^2 = \frac{1}{[c(x)]^2}$, so that $\frac{dt}{dx} = \frac{1}{c(x)}$ on ② and

$\frac{dt}{dx} = -\frac{1}{c(x)}$ on ③

Then (**) becomes (by (***))

$$0 = \int_{②} \frac{1}{2c^2} (w_t + c w_x)^2 \, dx + \int_{③} \frac{1}{2c^2} (w_t - c w_x)^2 \, dx + \frac{1}{2} (w(1,t_1))^2$$

Since all of the terms on the right hand side are nonnegative and continuous (or just a point, as in the case of $[w(1,t_1)]^2$), we must have, for all $(x,t) \in$ ② + ③

$$w_t + c w_x = 0 \qquad w_t - c w_x = 0 \qquad w(1,t_1) = 0$$

$$\begin{array}{r} w_t + c w_x = 0 \\ w_t - c w_x = 0 \\ \hline 2w_t = 0 \end{array} \Rightarrow w_t = 0 \Rightarrow w_x = 0$$

so, also $w_t(\bar{x}, t') = 0 \quad \forall 0 \le t' \le \bar{t} \Rightarrow \bar{t}$

These above statements lead to the proof of uniqueness $w(\bar{x},\bar{t}) = \int_0 w_t \ldots$

Now, the characteristics through $(\bar{x}, \bar{t})$ satisfy the differential $=0$ equations

②: $\dfrac{dt}{dx} = \dfrac{1}{c(x)}$ and ③: $\dfrac{dt}{dx} = -\dfrac{1}{c(x)}$

$$\Rightarrow \int_t^{\bar{t}} d\tau = \int_x^{\bar{x}} \frac{1}{c(\xi)}\, d\xi \qquad \Rightarrow \int_t^{\bar{t}} d\tau = \int_x^{\bar{x}} -\frac{1}{c(\xi)}\, d\xi$$

$$\bar{t} - t = \int_x^{\bar{x}} \frac{d\xi}{c(\xi)} \qquad \bar{t} - t = \int_{\bar{x}}^{x} \frac{d\xi}{c(\xi)}$$

$$\Rightarrow t = \bar{t} - \int_x^{\bar{x}} \frac{d\xi}{c(\xi)} \qquad \Rightarrow t = \bar{t} - \int_{\bar{x}}^{x} \frac{d\xi}{c(\xi)}.$$

$\square$