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Math 5440
HW3

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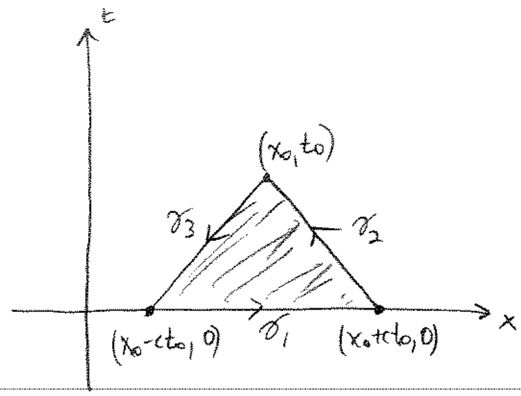

Let Ω denote the shaded triangle domain, and let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ be ~~the~~ its boundary. Suppose we are given the vector field $\vec{G} = \langle M, N \rangle = \langle u_t, c^2 u_x \rangle$. Green's Theorem states

$$\iint_\Omega (N_x - M_t)\, dA = \oint_{\partial\Omega} M dx + N dt = \int_a^b \vec{G}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt,$$

where $\vec{r}(t)$ is any parameterization of $\partial\Omega = \gamma$.

We begin with the right-hand side:

$$\oint_\gamma M dx + N dt = \oint_{\gamma_1} M dx + N dt + \oint_{\gamma_2} M dx + N dt + \oint_{\gamma_3} M dx + N dt.$$

For γ_1: A parameterization of γ_1 is $\vec{r}_1(t) = \langle x_0 - ct_0, 0 \rangle + \tau \langle 2ct_0, 0 \rangle$, $0 \le \tau \le 1$

So $$\oint_{\gamma_1} M dx + N dt = \int_0^1 \langle u_t(\vec{r}_1(t)), c^2 u_x(\vec{r}_1(t)) \rangle \cdot \langle 2ct_0, 0 \rangle\, d\tau$$

$$= \int_0^1 2ct_0\, u_t(x_0 - ct_0 + \tau 2ct_0, 0)\, d\tau$$

Let $v = x_0 - ct_0 + 2ct_0 \tau$
$dv = 2ct_0\, d\tau$

$$\Rightarrow \oint_{\gamma_1} M dx + N dt = \int_{x_0 - ct_0}^{x_0 + ct_0} 2ct_0\, u_t(v, 0) \frac{dv}{2ct_0}$$

$$= \int_{x_0 - ct_0}^{x_0 + ct_0} u_t(v, 0)\, dv$$

$$= \underline{\int_{x_0 - ct_0}^{x_0 + ct_0} g(v)\, dv}$$

For γ_2: A parameterization of γ_2 is:

$$\vec{r}_2(\tau) = \langle x_0 + ct_0, 0\rangle + \tau\langle -c, 1\rangle, \quad 0 \le \tau \le t_0$$

$$\Rightarrow \oint_{\gamma_2} M\,dx + N\,dt = \int_0^{t_0} \langle u_t(\vec{r}_2(\tau)), c^2 u_x(\vec{r}_2(\tau))\rangle \cdot \langle -c, 1\rangle \, d\tau$$

$$= \int_0^{t_0} \left[-c\,u_t(\vec{r}_2(\tau)) + c^2 u_x(\vec{r}_2(\tau))\right] d\tau$$

$$= -c \int_0^{t_0} \left[u_t(x_0 + ct_0 - c\tau, \tau) - c\,u_x(x_0 + ct_0 - c\tau, \tau)\right] d\tau$$

Now, $\frac{\partial}{\partial \tau} u(x_0 + ct_0 - c\tau, \tau) = -c u_x + u_t$

$$\Rightarrow \oint_{\gamma_2} M\,dx + N\,dy = -c \int_0^{t_0} \frac{\partial}{\partial \tau} u(x_0 + ct_0 - c\tau, \tau)\, d\tau$$

$$= -c\left[u(x_0 + ct_0 - ct_0, t_0) - u(x_0 + ct_0, 0)\right]$$

$$= \underline{-c\left[u(x_0, t_0) - f(x_0 + ct_0)\right]}$$

For γ_3: A parameterization of γ_3 is:

$$\vec{r}_3(\tau) = \langle x_0, t_0\rangle + \tau\langle -c, -1\rangle \qquad 0 \le \tau \le t_0$$

$$\Rightarrow \oint_{\gamma_3} M\,dx + N\,dt = \int_0^{t_0} \langle u_t(\vec{r}_3(\tau)), c^2 u_x(\vec{r}_3(\tau))\rangle \cdot \langle -c, -1\rangle \, d\tau$$

$$= \int_0^{t_0} \left[-c\,u_t(\vec{r}_3(\tau)) - c^2 u_x(\vec{r}_3(\tau))\right] d\tau$$

$$= c \int_0^{t_0} \left[-u_t(x_0 - c\tau, t_0 - \tau) - c\,u_x(x_0 - c\tau, t_0 - \tau)\right] d\tau$$

$$= c \int_0^{t_0} \frac{\partial}{\partial \tau} u(x_0 - c\tau, t_0 - \tau)\, d\tau$$

$$= c\, u(x_0 - c\tau, t_0 - \tau)\Big|_0^{t_0}$$

$$= c\left[u(x_0 - ct_0, 0) - u(x_0, t_0)\right]$$

$$= c\left[f(x_0 - ct_0) - u(x_0, t_0)\right]$$

Also, $\iint_\Omega (N_x - M_t)\,dA = \iint_\Omega (c^2 u_{xx} - u_{tt})\,dA = \iint_\Omega -F\,dA = -\iint_\Omega F\,dA$

Thus, by Greens Theorem,

$$\iint_\Omega (N_x - M_t)\,dA = \oint_{\partial\Omega} M\,dx + N\,dt$$

$$\Leftrightarrow \quad -\iint_\Omega F\,dA = \int_{x_0-ct_0}^{x_0+ct_0} g(v)\,dv - c\left[u(x_0,t_0) - f(x_0+ct_0)\right]$$

$$+ c\left[f(x_0-ct_0) - u(x_0,t_0)\right]$$

$$= c\left[f(x_0+ct_0) + f(x_0-ct_0)\right] - 2c\,u(x_0,t_0) + \int_{x_0-ct_0}^{x_0+ct_0} g(x)\,dx$$

$$\Rightarrow \quad 2c\,u(x_0,t_0) = c\left[f(x_0+ct_0) + f(x_0-ct_0)\right] + \int_{x_0-ct_0}^{x_0+ct_0} g(x)\,dx + \iint_\Omega F\,dA$$

$$\Rightarrow \quad u(x_0,t_0) = \frac{1}{2}\left[f(x_0+ct_0) + f(x_0-ct_0)\right] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(x)\,dx + \iint_\Omega F\,dA \quad \blacksquare$$

(2) Let $\Omega \subset \mathbb{R}^n$ be an open domain, $f: \Omega \to \mathbb{R}$ continuous. Suppose that for all open balls $B(x_0,r) = \{x \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| < r\}$ with $B(x_0,r) \subset \Omega$ we have $(*)$ $\int_{B(x_0,r)} f(x)\,dV = 0$. Prove $f(x) \equiv 0$ in Ω.

Proof: Assume for contradiction that $\exists x_0 \in \Omega$ s.t. $f(x_0) = \eta > 0$.
Since f is continuous, $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \Omega)(x \in B(x_0,\delta) \Rightarrow f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$
$= (\eta - \varepsilon, \eta + \varepsilon)$.

Suppose $\varepsilon = \eta/2$. Then $(\exists \delta > 0)(\forall x \in \Omega)\left(x \in B(x_0,\delta) \Rightarrow f(x) \in (\eta/2, 3\eta/2)\right)$, so $f(x) > \eta/2$ for all $x \in B(x_0,\delta)$. Hence

$$\int_{B(x_0,r)} f(x)\,dV \geq \int_{B(x_0,r)} \eta/2\,dV = \eta/2 \int_{B(x_0,r)} dV = \eta/2\,\mathrm{Vol}(B(x_0,r)) > 0. \quad \rightarrow\!\leftarrow$$

Now suppose $f(x_0) = -\eta$, where $\eta > 0$. Since f is continuous,

$$(\exists\, \delta > 0)\,(\forall x \in \Omega)\,\left(x \in B(x_0,\delta) \Rightarrow f(x) \in (-\eta - \eta/2, -\eta + \eta/2) = (-3\eta/2, -\eta/2)\right).$$

In particular if $x \in B(x_0, \delta)$, $f(x) < -\eta/2$. Thus

$$\int_{B(x_0,\delta)} f(x)\,dV \leq \int_{B(x_0,\delta)} -\eta/2 \, dV = -\eta/2 \,\mathrm{Vol}(B(x_0,\delta)) < 0 \quad \rightarrow\leftarrow .$$

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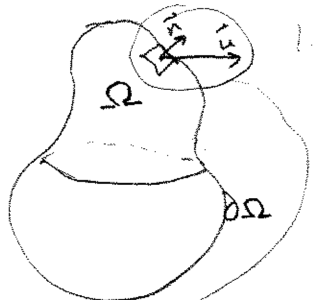
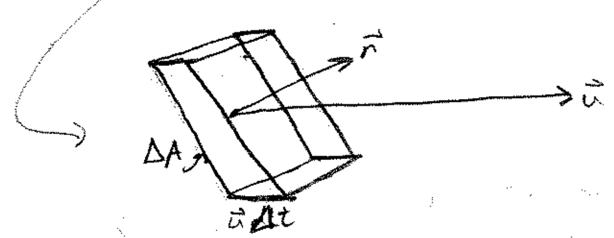
③ Derive the Navier-Stokes Equations!

$$\begin{cases} a) & \nabla\cdot(\rho\vec{u}) + \rho_t = 0 \\ b) & \rho\vec{u}_t + \rho\vec{u}\cdot\nabla\vec{u} + \nabla p = \rho\vec{F} \end{cases}$$

Soln: a) Consider a fixed domain $\Omega \subset \mathbb{R}^3$.



Let $\vec{u}(\vec{x},t)$ be the velocity field at $(\vec{x},t)$
$\rho(\vec{x},t)$ be the density $\left(\frac{\text{mass}}{\text{Vol}}\right)$ at $(\vec{x},t)$
and $(\vec{x},t)$ denote Eulerian coordinates.
In order to determine the mass of the fluid passing through $\partial\Omega$, consider a small "piece" of $\partial\Omega$ with outward normal $\vec{n}$. The total mass that passed through this piece of surface in time Δt, is given by the volume of this parallelepiped times the density, i.e.,



$$\Delta m = (\text{Volume})(\text{density})$$
$$= \Delta A (\vec{u}\,\Delta t)\cdot\vec{n}\,\rho$$

Dividing both sides by Δt and taking limits gives

$$\frac{dm}{dt} = \rho(\vec{u}\cdot\vec{n})\,dA = (\rho u)\cdot\vec{n}\,dA$$

The rate mass is going <u>into</u> Ω is given by

$$\frac{dm}{dt} = -\iint_{\partial\Omega} (\rho\vec{u})\cdot\vec{n}\,dA \underset{\substack{\uparrow \\ \text{divergence} \\ \text{theorem}}}{=} -\iiint_{\Omega} \nabla\cdot(\rho\vec{u})\,dV. \quad ①$$

On the other hand, the mass contained in Ω is given by:

$$m(t) = \iiint_{\Omega} \rho(\vec{x},t)\, dV$$

Therefore $$\frac{dm}{dt} = \frac{d}{dt}\iiint_{\Omega} \rho(\vec{x},t)\, dV$$

we can pass the derivative through the integrals, since Ω doesn't depend on t and we are integrating over the volume.

Thus $$\frac{dm}{dt} = \iiint_{\Omega} \rho_t(\vec{x},t)\, dV \quad (2)$$

Equating (1) and (2) we obtain the following:

$$-\iiint_{\Omega} \nabla\cdot(\rho\vec{u})\, dV = \iiint_{\Omega} \rho_t(\vec{x},t)\, dV$$

$$\Leftrightarrow \quad \iiint_{\Omega} \left[\rho_t + \nabla\cdot(\rho\vec{u})\right] dV = 0$$

This is true for any open domain $\Omega \subset \mathbb{R}^3$, in particular, for every open ball in $\mathbb{R}^3$. Hence #2 implies that

$$\boxed{\rho_t + \nabla\cdot(\rho\vec{u}) = 0}$$

b) We will now consider Largrangian coordinates $(\vec{x}(\vec{a},t),t)$. We have

$\frac{d}{dt}\vec{x}(\vec{a},t) = \vec{u}(\vec{x}(\vec{a},t),t)$ fluid velocity

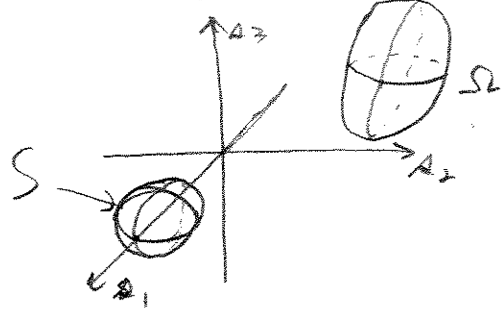
$\rho(\vec{x}(\vec{a},t),t)$ density (mass/volume)

$p(\vec{x}(\vec{a},t),t)$ pressure

$\vec{F}(\vec{x}(a,t),t)$ body forces.

We begin the case where the gas is at equilibrium, where we assume the density is constant and equal to ρ_0

Let S be a ball of radius R in $\mathbb{R}^3$, and let Ω be the region



containing the fluid "particles" that were originally in S at a later time. That is, we want to use Lagrangian coordinates, so we follow the individual motion of <u>each particle</u>.

Let $\vec{n}(\vec{x}(\vec{a},t),t)$ denote the unit normal vector to Ω at time t and position $\vec{x}(\vec{a},t)$.
Newton's 2nd Law states $\frac{d}{dt}$ (total momentum) $= \sum$ Forces.

The time derivative of the total momentum of the fluid is given by

$$\frac{d}{dt}\iiint_S \rho_0\, \vec{u}(\vec{x}(\vec{a},t),t)\, dV_s \,. \quad (3)$$

We can move the derivative through the integrals, since S does not depend on time and we are integrating over the volume and not over time. Thus (3) becomes

see below

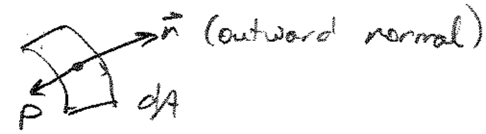
$$\iiint_S \rho_0 \frac{d}{dt}\vec{u}(\vec{x}(\vec{a},t),t)\, dV_s \overset{(*)}{=} \iiint_S \rho_0\left(\nabla\vec{u}\cdot\frac{\partial\vec{x}}{\partial t} + \vec{u}_t\right) dV_s$$

$$= \underline{\iiint_S \rho_0\left[(\nabla\vec{u})\cdot\vec{u} + \vec{u}_t\right] dV_s} \,.$$

(*) $$\frac{d}{dt}\vec{u}(\vec{x}(\vec{a},t),t) = \frac{d}{dt}\begin{bmatrix} u_1[x(\vec{a},t), y(\vec{a},t), z(\vec{a},t), t] \\ u_2[x(\vec{a},t), y(\vec{a},t), z(\vec{a},t), t] \\ u_3[x(\vec{a},t), y(\vec{a},t), z(\vec{a},t), t] \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u_1}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u_1}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u_1}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u_2}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u_2}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u_2}{\partial t} \\ \frac{\partial u_3}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u_3}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u_3}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u_3}{\partial t} \end{bmatrix} = \begin{bmatrix} \nabla u_1\cdot\frac{\partial\vec{x}}{\partial t} \\ \nabla u_2\cdot\frac{\partial\vec{x}}{\partial t} \\ \nabla u_3\cdot\frac{\partial\vec{x}}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \\ \frac{\partial u_3}{\partial t} \end{bmatrix} = \nabla\vec{u}\cdot\vec{u} + \vec{u}_t$$

The net forces acting on the fluid are due to pressure and external forces. If we look at a small "piece" of $\partial\Omega$, we have



Thus the force due to pressure on Ω is:

$$\iint_{\partial\Omega} -p\vec{n}\, dA$$

The external forces are given by

$$\iiint_{S} \rho_0 \vec{F}(\vec{x}(\vec{a},t),t)\, dV_s$$

Thus the net force acting on the fluid is

$$\sum F = -\iint_{\partial\Omega} p\vec{n}\, dA + \underline{\iiint_{S} \rho_0 \vec{F}(\vec{x}(\vec{a},t),t)\, dV_s}$$

Now,

$$\iint_{\partial\Omega} p\vec{n}\, dA = \begin{bmatrix} \iint_{\partial\Omega} p n_1\, dA \\ \iint_{\partial\Omega} p n_2\, dA \\ \iint_{\partial\Omega} p n_3\, dA \end{bmatrix} = \begin{bmatrix} \iint_{\partial\Omega} p\vec{e}_1\cdot\vec{n}\, dA \\ \iint_{\partial\Omega} p\vec{e}_2\cdot\vec{n}\, dA \\ \iint_{\partial\Omega} p\vec{e}_3\cdot\vec{n}\, dA \end{bmatrix} \overset{\text{divergence theorem}}{=} \begin{bmatrix} \iiint_{\Omega} \nabla\cdot p\vec{e}_1\, dV \\ \iiint_{\Omega} \nabla\cdot p\vec{e}_2\, dV \\ \iiint_{\Omega} \nabla\cdot p\vec{e}_3\, dV \end{bmatrix}$$

$$= \begin{bmatrix} \iiint_{\Omega} p_x\, dV \\ \iiint_{\Omega} p_y\, dV \\ \iiint_{\Omega} p_z\, dV \end{bmatrix} = \underline{\iiint_{\Omega} \nabla p(\vec{x}(\vec{a},t),t)\, dV}$$

Thus Newton's 2nd Law implies

$$\iiint_{S} \rho_0[\vec{u}\cdot(\nabla\vec{u}) + \vec{u}_t]\, dV_s = -\iiint_{\Omega} \nabla p\, dV + \iiint_{S} \rho_0 \vec{F}\, dV_s \quad (4)$$

The final step is to change coordinates from S to Ω in the integral on the left and the last integral on the right. When we do this, we have $\rho_0 \, dV_s = \rho(\vec{x}(\vec{a},t),t)\, dV = \rho J dV_s$, where J denotes the determinant of the Jacobian of the transformation from S to Ω. This implies that $\rho_0 = \rho J$. Thus ④ becomes

$$\iiint_\Omega (\vec{u}\cdot\nabla\vec{u} + \vec{u}_t)\rho \, dV = -\iiint_\Omega \nabla p \, dV + \iiint_\Omega \vec{F}\rho \, dV$$

$$\Leftrightarrow \iiint_\Omega \left[(\vec{u}_t + \vec{u}\cdot\nabla\vec{u})\rho + \nabla p - \rho\vec{F}\right] dV = 0$$

Since Ω was an arbitrary domain in $\mathbb{R}^3$, Problem ② implies

$$\rho\vec{u}_t + \rho\vec{u}\cdot\nabla\vec{u} + \nabla p - \rho\vec{F} = 0$$

$$\Leftrightarrow \boxed{\rho\vec{u}_t + \rho\vec{u}\cdot\nabla\vec{u} + \nabla p = \rho\vec{F}.}$$

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④ $$\vec{E} = \frac{c\vec{r}}{\|\vec{r}\|^3} = \frac{c\langle x,y,z\rangle}{(x^2+y^2+z^2)^{3/2}}$$

a) By computing the constant value of $\vec{E}\cdot\vec{n}$ of the sphere of radius R deduce that the flux integral

$$\iint_{\|\vec{r}\|=R} \vec{E}\cdot\vec{n} = 4\pi c$$

independently of R.

b) Compute $\operatorname{div}(\vec{E}) = \nabla\cdot\vec{E}$ and show it is 0 for all $\vec{x} \neq \vec{0}$.

c) Consider a sum of translations and scalings of the vector field $\vec{E}$:

$$\vec{F} = \frac{c_1(\vec{r}-\vec{r}_1)}{\|\vec{r}-\vec{r}_1\|^3} + \cdots + \frac{c_k(\vec{r}-\vec{r}_k)}{\|\vec{r}-\vec{r}_k\|^3}$$

Use the divergence thm, linearity of divergence and flux integrals w/r/t vector fields to show if Ω is any 3-d domain

(with piecewise smooth $\partial\Omega$, so div. thm. applies), then
$$\iint_{\partial\Omega} \vec{F}\cdot\vec{n}\, dS = 4\pi(c_1 + \cdots + c_k).$$

Soln:

a) For a sphere, the unit exterior normal is $\frac{\vec{r}}{\|\vec{r}\|}$. Thus
$$\vec{E}\cdot\vec{n} = \frac{c\vec{r}}{\|\vec{r}\|^3}\cdot\frac{\vec{r}}{\|\vec{r}\|} = \frac{c(\vec{r}\cdot\vec{r})}{\|\vec{r}\|^4} = \frac{c\|\vec{r}\|^2}{\|\vec{r}\|^4} = \frac{c}{\|\vec{r}\|^2} = \frac{c}{R^2}$$

Thus $\displaystyle\iint_{\|\vec{r}\|=R} \vec{E}\cdot\vec{n}\, dA = \iint_{\|\vec{r}\|=R} \frac{c}{R^2}\, dA = \frac{c}{R^2}\iint_{\|\vec{r}\|=R} dA = \frac{c}{R^2}\cdot 4\pi R^2 = \boxed{4\pi c}$

b) $\nabla\cdot\vec{E} = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}$.

Now, $E_1 = \frac{cx}{(x^2+y^2+z^2)^{3/2}}$, so
$$\frac{\partial E_1}{\partial x} = c\,\frac{(x^2+y^2+z^2)^{3/2} - x\cdot\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2}\cdot\cancel{2}x}{(x^2+y^2+z^2)^3} = c\,\frac{\|\vec{r}\|^3 - 3x^2\|\vec{r}\|}{\|\vec{r}\|^6}$$
$$= c\,\frac{\|\vec{r}\|^2 - 3x^2}{\|\vec{r}\|^5}.$$

By symmetry, $\frac{\partial E_2}{\partial y} = c\,\frac{\|\vec{r}\|^2 - 3y^2}{\|\vec{r}\|^5}$ and $\frac{\partial E_3}{\partial z} = c\,\frac{\|\vec{r}\|^2 - 3z^2}{\|\vec{r}\|^5}$,

so $$\nabla\cdot\vec{E} = c\,\frac{\|\vec{r}\|^2 - 3x^2}{\|\vec{r}\|^5} + c\,\frac{\|\vec{r}\|^2 - 3y^2}{\|\vec{r}\|^5} + c\,\frac{\|\vec{r}\|^2 - 3z^2}{\|\vec{r}\|^5}$$
$$= c\,\frac{3\|\vec{r}\|^2 - 3(x^2+y^2+z^2)}{\|\vec{r}\|^5} = c\,\frac{3\|\vec{r}\|^2 - 3\|\vec{r}\|^2}{\|\vec{r}\|^5} = 0 \text{ for } \vec{r}\neq 0.$$

c) We first note that $\nabla\cdot\vec{F} = \nabla\cdot\sum_{j=1}^{k}\vec{E}_j = \sum_{j=1}^{k}\nabla\cdot\vec{E}_j$, where
$$\vec{E}_j = \frac{c_j(\vec{r}-\vec{r}_j)}{\|\vec{r}-\vec{r}_j\|^3}, \quad j = 1, 2, \ldots, k.$$

By part b), $\nabla \cdot \vec{E}_j = 0$ everywhere except at $\vec{r}_j$ because, if we let $\vec{u}_j = \vec{r} - \vec{r}_j$, we have

$$\vec{E}_j = \frac{c_j \vec{u}_j}{\|\vec{u}_j\|^3}, \quad \text{and} \quad \nabla \cdot \vec{E}_j = 0 \text{ except where } \vec{u}_j = 0 \Leftrightarrow \vec{r} - \vec{r}_j = 0 \Leftrightarrow \vec{r} = \vec{r}_j.$$

Thus $\displaystyle\iiint_\Omega \nabla \cdot \vec{F}\, dV = \iiint_\Omega \sum_{j=1}^{k} \nabla \cdot \vec{E}_j \, dV = \sum_{j=1}^{k} \iiint_\Omega \nabla \cdot \vec{E}_j \, dV.$

Now, we know that $\nabla \cdot \vec{E}_j = 0$ if $\vec{r} \neq \vec{r}_j$. Thus $\displaystyle\iiint_\Omega \nabla \cdot \vec{E}_j \, dV$ can be found by integrating over a small ball around $\vec{r}_j$, as long as this ball is contained within Ω. Choose $\varepsilon > 0$ so that $B(\vec{r}_j, \varepsilon) \cap B(\vec{r}_i, \varepsilon) = \emptyset$ for $i, j = 1, \ldots, k$ where $i \neq j$ and so that $B(\vec{r}_j, \varepsilon) \subset \Omega$ for $j = 1, \ldots, k$. (This is assuming that all of the $\vec{r}_j$'s are in Ω.) Then

$$\iiint_\Omega \nabla \cdot \vec{F}\, dV = \sum_{j=1}^{k} \iiint_\Omega \nabla \cdot \vec{E}_j \, dV = \sum_{j=1}^{k} \iiint_{B(\vec{r}_j, \varepsilon)} \nabla \cdot \vec{E}_j \, dV$$

By part a), $\displaystyle\iiint_{B(\vec{r}_j, \varepsilon)} \nabla \cdot \vec{E}_j = 4\pi c_j.$

Therefore $\displaystyle\iiint_\Omega \nabla \cdot \vec{F}\, dV = \sum_{j=1}^{k} \iiint_{B(\vec{r}_j, \varepsilon)} \nabla \cdot \vec{E}_j = \sum_{j=1}^{k} 4\pi c_j = 4\pi \sum_{j=1}^{k} c_j.$ (5)

By the divergence theorem, $\displaystyle\iiint_\Omega \nabla \cdot \vec{F}\, dV = \iint_{\partial\Omega} \vec{F} \cdot \vec{n}\, dS$

Thus, by (5), $\displaystyle\iint_{\partial\Omega} \vec{F} \cdot \vec{n}\, dS = 4\pi \sum_{j=1}^{k} c_j = 4\pi (c_1 + \cdots + c_k).$ ∎