Math 5440
HW due 8/27; HW #1

1. Derive (1.13)
$$\begin{cases} \rho u_x + u\rho_x + \rho_t = 0 \\ \rho u_t + \rho u u_x + p_x = \rho F \end{cases}$$

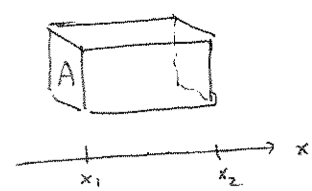
plane wave
propogation of disturbances in friction-free fluids like gases, in the x-direction.

(all fcns constant in y, z directions)

$\rho(x,t)$ = density mass/volume
$p(x,t)$ = pressure force/area acts in normal direction on surfaces
$F(x,t)$ = body forces/mass (so ρF = body forces/volume).

1a
First eqtn is conservation of mass:
consider the total mass in a fixed box of y-z sectional area A, extending from $x_1 \le x \le x_2$

$$M(t) := \int_{x_1}^{x_2} \rho(x,t) A\, dx$$



mass leaves box through the left & right faces (or enters).
if x-velocity @ $x=x_1$ is $u(x_1,t)$ then in time Δt approximately a mass of $\underbrace{A u(x_1,t)\Delta t}_{vol}\, \rho(x_1,t)$ enters @ left face.

so is entering at a rate of $A u(x_1,t)\rho(x_1,t)$ $\left(\lim_{\Delta t\to 0} \frac{\Delta M}{\Delta t} \text{ thru left face}\right)$
Also leaving @ rate $A u(x_2,t)\rho(x_2,t)$ @ right face.

Thus $M'(t)$ = flux of $u\vec{e}_1$ into box
$$(*) \qquad = A u\rho\Big]_{(x_2,t)}^{(x_1,t)}$$

In HW 3.3 due 9/10 also use a flux surface integral for M'(t) in any test domain

Alternately,
$$(**) \qquad M'(t) = \frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t) A\, dx = \int_{x_1}^{x_2} \rho_t A\, dx.$$

Equate (*), (**), cancel A, deduce
$$u\rho\Big]_{(x_2,t)}^{(x_1,t)} = \int_{x_1}^{x_2} \rho_t\, dx$$

alternate path: Use FTC backwards on left side to get
$$\int_{x_1}^{x_2} (u\rho)_x\, dx = \int_{x_1}^{x_2} \rho_t\, dx$$

take $\frac{d}{dx_2}$: $-\frac{\partial}{\partial x_2}(u\rho)(x_2,t) = \rho_t(x_2,t)$ true $\forall x_2$

replace $x_2 \to x$ and use product rule:

$$-u_x\rho - u\rho_x = \rho_t$$

i.e. $\boxed{\rho u_x + \rho_x u + \rho_t = 0}$

since this is true for all intervals, and since all fcns are diff'ble (assumed) so these integrands are continuous deduce
$$(u\rho)_x = \rho_t$$

You'll use this path in HW #3.3 due 9/10

(1b) Newton's 2^{nd} law

In this case it's easier to use a box moving with the fluid particles. (We would've done this in 1a) too, but a rest frame box was easier there.)

Thus, we're considering plane wave particles whose x-coordinates are given by $x(s,t)$ where s is a fixed background x-direction variable

path $x(s,t)$ of plane wave face $\perp$ to x-axis

s $\quad x$

it satisfies $x_t = u(x(s,t),t)$

particle velocity in x-dir.

unknown velocity field in x-dir, evaluated at particle location for given time

We'll study $p(x(s,t),t)$, $\rho(x(s,t),t)$ in addition to $u(x(s,t),t)$.

The convention is to use special notation for $\frac{\partial}{\partial t}$ of these composition functions, namely $\frac{D}{Dt}$, and to call these derivatives $\boxed{\text{material derivatives}}$

(See wiki page on derivation of Navier Stokes)

i.e. $\frac{D}{Dt}p = p_x x_t + p_t = p_x u + p_t$ (for actual 3-d motion this becomes $\frac{D}{Dt}p = \nabla p \cdot \vec{u} + \frac{\partial p}{\partial t}$.)

Now: moving box:

box moving & stretching in x dir

right face moving with x-velocity $x_t = u(x(s_2,t),t)$

left face w x-coord at $x(s_1,t)$

area A

$x_t = u(x(s,t),t)$

$s_1 \quad s_2 \quad x$

Newton: $\frac{d}{dt}$ (momentum in fluid box) = net forces.

$$\frac{d}{dt}\left[\int_{s_1}^{s_2} A\rho_0 \, u \, ds\right] = pA\Big]_{(s_2,t)}^{(s_1,t)} + \int_{s_1}^{s_2} A\rho_0 F \, ds$$

const equil density at rest

$u(x(s,t),t)$

$p(x(s,t),t)$

pressure pushes in + dir on left, in − dir on right.

$F(x(s,t),t)$ body force/unit mass.

for a general test domain, pressure is assumed to create a force on the boundary surface, in the inner normal direction.

pass $\frac{d}{dt}$ thru integral & remember the chain rule for material derivatives: Also convert 1st term on right to an int. using FTC

$$\int_{s_1}^{s_2} A\rho_0 (u_x u + u_t)\,ds = -\int_{s_1}^{s_2} A \underbrace{\frac{d}{ds} p(x(s,t),t)}_{p_x x_s}\,ds + \int_{s_1}^{s_2} A\rho_0 F\,ds$$

$$A\int_{s_1}^{s_2} \rho_0(u_x u + u_t) + p_x x_s - \rho_0 F \; ds = 0$$

true for all test intervals $[s_1, s_2]$, so

(alternate: take $\frac{d}{ds_2}$ of integral expression).

$$* \quad \rho_0(u_x u + u_t) + p_x x_s - \rho_0 F = 0$$

now, by conservation of mass,

$$\rho_0 ds = \rho\,|x_s|\,ds \quad \longleftarrow$$

slab of thickness ds is stretched to thickness $|x_s|ds$ cross-sectional area stays the same.

$$= \rho\, x_s\, ds$$

since $x(s,t) \approx s$
$x_s \approx 1 > 0$.

so $\boxed{\rho = \frac{\rho_0}{x_s}}$ $\longleftarrow$ in general wave motion, the denominator will be a Jacobian determinant.

Thus if you divide $*$ by x_s you get the desired result

$$\boxed{\rho u_x u + \rho u_t + p_x = \rho F}$$

2. Linearization: $\rho = \rho_0 + O(\varepsilon)$, also ρ_x, ρ_t
$u = O(\varepsilon)$, also u_x, u_t
$p = p_0 + O(\varepsilon)$, also p_x, p_t $O(\varepsilon)$
$F = O(\varepsilon)$

I forgot a couple of these assumptions when I assigned the problem

$$(1.13)\quad \begin{cases} (\rho_0 + O(\varepsilon))\,u_x + u\cdot O(\varepsilon) + \rho_t = 0 \\ (\rho_0 + O(\varepsilon))\,u_t + (\rho_0 + O(\varepsilon))\,O(\varepsilon)\cdot O(\varepsilon) + p_x = (\rho_0 + O(\varepsilon))\,F \end{cases}$$

$\downarrow$ discard $O(\varepsilon^2)$ terms

$$(1.14)\quad \begin{cases} \rho_0 u_x + \rho_t = 0 & (a) \\ \rho_0 u_t + p_x = \rho_0 F & (b) \end{cases}$$

if $p = p(\rho)$ i.e. only (x,t) dependence is thru $\rho(x,t)$

then $p_x = p'(\rho)\rho_x$ Since $p'(\rho) = p'(\rho_0) + O(\varepsilon)$ (since $\rho - \rho_0 = O(\varepsilon)$)

we get $p_x = (p'(\rho_0) + O(\varepsilon))\,\rho_x = p'(\rho_0)\rho_x + O(\varepsilon^2)$

so replace p_x with $p'(\rho_0)\rho_x$

3 Wave eqtns for ρ & u:

1.14a $\quad \rho_0 u_x + \rho_t = 0$

1.14b $\quad \rho_0 u_t + p'(\rho_0)\rho_x = \rho_0 F$

$\partial_t(1.14a)$: $\quad \rho_0 u_{xt} + \rho_{tt} = 0$

$\partial_x(1.14b)$: $\quad \rho_0 u_{tx} + p'(\rho_0)\rho_{xx} = \rho_0 F_x$

$\partial_t(1.14a) - \partial_x(1.14b)$: $\boxed{\rho_{tt} - p'(\rho_0)\rho_{xx} = -\rho_0 F_x}$

$\partial_x(1.14a)$: $\quad \rho_0 u_{xx} + \rho_{tx} = 0$

$\partial_t(1.14b)$ $\quad \rho_0 u_{tt} + p'(\rho_0)\rho_{xt} = \rho_0 F_t$

$\frac{1}{\rho_0}\partial_t(1.14b) - \frac{p'(\rho_0)}{\rho_0}\partial_x(1.14a)$:

$$\boxed{u_{tt} - p'(\rho_0)u_{xx} = F_t}$$

4 $\quad pV^\gamma = C \qquad C = C(T, m)$, $\gamma \approx 1.4$ for air

abs. temp

fixed mass.

$p = \frac{C}{V^\gamma}$

since $\rho = \frac{m}{V}$,

$p = C_1 \rho^\gamma \qquad C_1 = \frac{C}{m^\gamma}$

$\Rightarrow p'(\rho) = C_1 \gamma \rho^{\gamma-1}$

$= C_1 \rho^\gamma \gamma \rho^{-1}$

$$\boxed{p'(\rho) = p\gamma\rho^{-1}}$$

according to wikipedia page
/wiki/Density_of_air

@ 25° C, $c = 346.18$ m/sec

$\rho = 1.184$ kg/m³ (probably at sea level)

p = not mentioned, but assuming it is 1 atmosphere,

1 atm $= \frac{100}{.986}$ kPA (10^3 N/m)

$= 1.013 \times 10^5$ N/m

(according to on-line converter)

so $\quad c^2 = p\gamma\rho^{-1}$

$= \frac{1.013 \times 10^5 \times 1.4}{1.184}$

$\approx 1.20 \times 10^5$

$\Rightarrow c \approx \sqrt{1.2 \times 10^5} \approx 346$ m/sec !