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Math 5440

6.31.3: $f(x,y) = (1-y^2)\sin x \quad 0<x<\pi \quad 0<y<\pi$

$f(x,y) \sim \sum_{n,m=1}^{\infty} d_{mn} \sin my \sin nx$

$d_{mn} = \frac{2}{\pi^2}\int_0^\pi \int_{-\pi}^{\pi} (1-y^2)\sin x \sin nx \sin my \, dx\, dy$

$= \frac{2}{\pi^2}\underbrace{\left[\int_0^\pi (1-y^2)\sin my\, dy\right]}_{A_m}\underbrace{\left[\int_{-\pi}^{\pi}\sin x \sin nx\, dx\right]}_{B_n}$

$A_m = \int_0^\pi (1-y^2)\sin my\, dy = \left[\frac{-(1-y^2)\cos my}{m} - \frac{2y\sin my}{m^2} - \frac{2\cos my}{m^3}\right]_0^\pi$

$= -\frac{(1-\pi^2)\cos(m\pi)}{m} - \frac{2\pi \sin(m\pi)}{m} - \frac{2\cos(m\pi)}{m^3} + \frac{(1)\cos(0)}{m} + \frac{2(0)\sin(0)}{m^2} + \frac{2\cos(0)}{m^3}$

$= \frac{1}{m}\left((\pi^2-1)(-1)^m + 1\right) + \frac{2}{m^3}\left(1-(-1)^m\right)$

$B_n = \begin{cases} 1 & n=1 \\ 0 & n \neq 0 \end{cases}$

$\Rightarrow f(x,y) \sim \sum_{m=1}^{\infty}\left[\frac{2}{\pi m}\left(1+(-1)^m(\pi^2-1)\right) + \frac{4}{\pi m^3}\left(1-(-1)^m\right)\right]\sin x \sin my$

6.32.1:

$\nabla^2 u = 0 \quad x\in(0,\pi) \quad y\in(0,\pi) \quad z\in(0,\pi)$

$u = 0 \quad x=0, x=\pi, y=0, y=\pi, z=\pi$

$u(x,y,0) = \sin x \sin^3 y$

$u = X(x)\cdot Y(y)\cdot Z(z) \Rightarrow \nabla^2 u = X''YZ + Y''XZ + Z''XY = 0$

$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = \frac{-Z''}{Z} = C_1 \quad Z(0)=1,\ Z(\pi)=0 \quad Z'' = -C_1 Z$

$\frac{X''}{X} = C_1 - \frac{Y''}{Y} = C_2 \quad X(0)=X(\pi)=0 \quad X'' = C_2 X \Rightarrow C_2 = -n^2$

$\frac{Y''}{Y} = C_1 - C_2 \quad Y(0) = Y(\pi) = 0 \quad Y'' = (C_1 - C_2)Y \Rightarrow C_1 - C_2 = -m^2$

$X = \sin nx \quad Y = \sin my$

$C_1 = -m^2 - n^2 \Rightarrow Z = \alpha_{mn}\sinh\left(\sqrt{m^2+n^2}\,(\pi - z)\right)$

$u = \sum_1^\infty\sum_1^\infty \alpha_{nm}\sinh\left(\sqrt{m^2+n^2}\,(\pi-z)\right)\sin nx \sin my$

$g(x,y) = u(x,y,0) = \sum_1^\infty\sum_1^\infty \alpha_{nm}\sinh\left(\sqrt{m^2+n^2}\,\pi\right)\sin nx \sin my$

$\alpha_{nm}\sinh\left(\sqrt{m^2+n^2}\,\pi\right) = d_{nm} \equiv \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \sin x \sin^3 y \sin nx \sin my$

$d_{nm} = \frac{4}{\pi^2}\underbrace{\left[\int_{-\pi}^{\pi}\sin x \sin nx\, dx\right]}_{A_n}\underbrace{\left[\int_0^\pi \sin^3 y \sin my\, dy\right]}_{B_n}$

$A_n = \begin{cases} 1 & n=1 \\ 0 & n\neq 1 \end{cases} \qquad B_n = \begin{cases} 3/4 & n=1 \\ -1/4 & n=3 \\ 0 & n\neq 1,3 \end{cases}$

$\alpha_{nm} = \frac{d_{nm}}{\sinh\left(\sqrt{m^2+n^2}\,\pi\right)}$

$$u(x,y,z) = \frac{3}{4}\sin x \sin y \frac{\sinh\sqrt{2}\,(\pi - z)}{\sinh\sqrt{2}\,\pi} - \frac{1}{4}\sin x \sin 3y \frac{\sinh\sqrt{10}\,(\pi - z)}{\sinh\sqrt{10}\,\pi}$$

6.32.4

$U_t - U_{xx} - U_{yy} = 0 \quad x \in (0,\pi) \quad y \in (0,\pi) \quad t > 0$

$u = 0 \quad x = 0,\ x = \pi,\ y = 0,\ y = \pi$

$u(x,y,0) = f(x,y)$

$u = X(x)\,Y(y)\,T(t) \Rightarrow \frac{T'}{T} - \frac{X''}{X} - \frac{Y''}{Y} = 0$

$\frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = C_1 \qquad T' - C_1 T = 0$

$\frac{X''}{X} = C_1 - \frac{Y''}{Y} = C_2 \qquad X'' - C_2 X = 0 \Rightarrow C_2 = -n^2$

$\frac{Y''}{Y} = C_1 - C_2 \qquad Y'' + (C_2 - C_1)Y = 0 \Rightarrow C_2 - C_1 = m^2 \Rightarrow C_1 = C_2 - m^2 = -(n^2+m^2)$

$X = \sin(nx) \quad Y = \sin(my)$

$Z = \alpha_{nm} e^{-(n^2+m^2)t}$

$u(x,y,t) = \sum_1^\infty \sum_1^\infty \alpha_{nm} e^{-(n^2+m^2)t} \sin nx \sin my$

$u(x,y,0) = g(x,y) = \sum_1^\infty \sum_1^\infty \alpha_{nm} \sin nx \sin my$

$\alpha_{nm} = d_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y) \sin nx \sin my \, dx\, dy$

$$u(x,y,t) = \sum_1^\infty \sum_1^\infty d_{nm} e^{-(n^2+m^2)t} \sin nx \sin mx$$

6.32.5

$U_{tt} - U_{xx} - U_{yy} = 0 \quad 0 < x < \pi,\ 0 < y < A \quad t > 0$

$u = 0 \quad x = 0, \pi, \quad y = 0, A \qquad u = X(x)\,Y(y)\,T(t) \Rightarrow \frac{T''}{T} - \frac{X''}{X} - \frac{Y''}{Y} = 0$

$u(x,y,0) = f(x,y) \qquad \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = C_1$

$u_t(x,y,0) = g(x,y) \qquad \frac{X''}{X} = C_1 - \frac{Y''}{Y} = C_2 \qquad X = \sin nx$

$X'' = C_2 X \quad C_2 = -n^2 \qquad Y = \sin\left(\frac{m\pi}{A} y\right)$

$Y'' = (C_1 - C_2)Y \Rightarrow C_1 - C_2 = -\lambda^2 \Rightarrow C_1 = -\lambda^2 + C_2 = -(\lambda^2 + n^2)$

$T = T_1 + T_2 \qquad u_1(x,y,0) = f(x,y) \qquad u_{2t}(x,y,0) = g(x,y)$

$T'' = -\left(\frac{m^2\pi^2}{A^2} + n^2\right)T \Rightarrow T_1 = \alpha_{mn} \cos\left(\sqrt{\frac{m^2\pi^2}{A^2} + n^2}\, t\right) \quad T_2 = \beta_{mn} \sin\left(\sqrt{\frac{m^2\pi^2}{A^2} + n^2}\, t\right)$

$u(x,y,t) = \sum_1^\infty \sum_1^\infty \left[\alpha_{mn} \cos\left(\sqrt{n^2 + \frac{m^2\pi^2}{A^2}}\, t\right) + \beta_{mn} \sin\left(\sqrt{\frac{m^2\pi^2}{A^2} + n^2}\, t\right)\right] \sin nx \sin my$

$\alpha_{mn} \cos(0) = d_{mn} = \frac{4}{\pi A}\int_0^\pi\int_0^A f(x,y) \sin\left(\frac{m\pi}{A} y\right) \sin(nx)\, dy\, dx$

$\sqrt{\frac{m^2\pi^2}{A^2} + n^2}\ \beta_{mn} \cos(0) = e_{nm} = \frac{4}{\pi A}\int_0^\pi\int_0^A g(x,y) \sin\left(\frac{m\pi}{A} y\right) \sin(nx)\, dy\, dx$

$$u(x,y,t) = \sum_1^\infty \sum_1^\infty \left[d_{mn} \cos\left(\sqrt{n^2 + \frac{m^2\pi^2}{A^2}}\, t\right) + \frac{e_{nm}}{\sqrt{\frac{m^2\pi^2}{A^2} + n^2}} \sin\left(\sqrt{n^2 + \frac{m^2\pi^2}{A}}\, t\right)\right] \sin\left(\frac{m\pi}{A} y\right) \sin(nx)$$

Class Exercise 1:

a. $u(\vec{x},t) = R(r)\,\Theta(\theta)\,T(t)$ $\quad \Delta u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\theta^2 u$

$\Delta u = R'\Theta T + rR''\Theta T + \frac{1}{r^2}RT\Theta''$

$u_t - k\Delta u = 0 = T'R\Theta - k\left[\frac{1}{r}R'\Theta T + R''\Theta T + \frac{1}{r^2}RT\Theta''\right] = 0$

$0 = \frac{T'}{T} - \frac{k}{r}\frac{R'}{R} - k\frac{R''}{R} - \frac{k}{r^2}\frac{\Theta''}{\Theta} \Rightarrow \frac{T'}{T} = \frac{k}{r}\frac{R'}{R} + k\frac{R''}{R} + \frac{k}{r^2}\frac{\Theta''}{\Theta} = \lambda$

Case 1 $\lambda = 0$: $T' = 0 \Rightarrow T = A_0$

Case 2 $\lambda > 0$: $T' = m^2 T \Rightarrow T = A_0 e^{m^2 t}$

Case 3 $\lambda < 0$: $T' = -m^2 T \Rightarrow T = A_0 e^{-m^2 t}$

$\frac{k}{r}\frac{R'}{R} + k\frac{R''}{R} = \frac{-k}{r^2}\frac{\Theta''}{\Theta} + \lambda \Rightarrow \frac{\Theta''}{\Theta} = -r\frac{R'}{R} - r^2\frac{R''}{R} + \frac{r^2}{k}\lambda = \delta$

$\delta < 0$ $\quad \Theta'' = -n^2\Theta$ $\quad \Theta = B\cos(n\theta) + C\sin(n\theta)$ $\quad (n \geq 1)$

$n = 0 \quad \Theta = C$

$-r\frac{R'}{R} - r^2\frac{R''}{R} + \frac{r^2}{k}\lambda = \delta$, nasty ODE.

b. $u_{tt} - c^2\Delta u = 0 = T''R\Theta - c^2\left[\frac{1}{r}R'\Theta T + R''\Theta T + \frac{1}{r^2}RT\Theta''\right]$

$0 = \frac{T''}{T} - \frac{c^2}{r}\frac{R'}{R} - c^2\frac{R''}{R} - \frac{c^2}{r^2}\frac{\Theta''}{\Theta} \Rightarrow \frac{T''}{T} = \frac{c^2}{r}\frac{R'}{R} + c^2\frac{R''}{R} + \frac{c^2}{r^2}\frac{\Theta''}{\Theta} = \lambda$

Case 1 $\lambda = 0$ $\quad T'' = 0 \Rightarrow T = At + B$

Case 2 $\lambda < 0$: $T'' = -n^2 T \Rightarrow T = A\cos(nt) + B\sin(nt)$

Case 3 $\lambda < 0$ $\quad T'' = n^2 T \Rightarrow T = A\cosh(nt) + B\sinh(nt)$

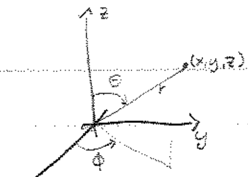
$-\frac{c^2}{r}\frac{R'}{R} - c^2\frac{R''}{R} + \lambda = \frac{\Theta''}{\Theta}\cdot\frac{c^2}{r^2} \Rightarrow \frac{\Theta''}{\Theta} = -r\frac{R'}{R} - r^2\frac{R''}{R} + \frac{r^2}{c^2}\lambda = \delta$

$\delta < 0 \Rightarrow \Theta'' = -m^2\Theta \Rightarrow \Theta = C\cos mt + D\sin mt$ $\quad m \geq 1$

$m = 0 \Rightarrow \Theta = C$

$-r\frac{R'}{R} - r^2\frac{R''}{R} + \frac{r^2}{c^2}\lambda = \delta$

Class Exercise 2:



$r=\sqrt{x^2+y^2+z^2}$ $\quad\theta=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$ $\quad\phi=\tan^{-1}(y/x)$

$\Delta=\partial_x^2+\partial_y^2+\partial_z^2$

$\partial_x=r_x\partial_r+\theta_x\partial_\theta+\phi_x\partial_\phi \qquad \partial_x^2=r_{xx}\partial_r+(r_x)^2\partial_r^2+\theta_{xx}\partial_\theta+(\theta_x)^2\partial_\theta^2+\phi_{xx}\partial_\phi+(\phi_x)^2\partial_\phi^2$

$\partial_y=r_y\partial_r+\theta_y\partial_\theta+\phi_y\partial_\phi \qquad \partial_y^2=r_{yy}\partial_r+(r_y)^2\partial_r+\theta_{yy}\partial_\theta+(\theta_y)^2\partial_\theta^2+\phi_{yy}\partial_\phi+(\phi_y)^2\partial_\phi^2$

$\partial_z=r_z\partial_r+\theta_z\partial_\theta+\phi_z\partial_\phi \qquad \partial_z^2=r_{zz}\partial_r+(r_z)^2\partial_r+\theta_{zz}\partial_\theta+(\theta_z)^2\partial_\theta^2+\phi_{zz}\partial_\phi+(\phi_z)^2\partial_\phi^2$

$r_x=\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}=\frac{x}{r}=\sin\theta\cos\phi \qquad r_y=\frac{y}{r}=\sin\theta\sin\phi \qquad r_z=\frac{z}{r}=\cos\theta$

$\theta_x=\frac{\partial}{\partial x}\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)=\frac{1}{\frac{x^2+y^2}{z^2}+1}\cdot\frac{1}{2}\cdot\frac{(x^2+y^2)^{1/2}}{z}\cdot 2x=\frac{zx}{r^2(x^2+y^2)^{1/2}}=\frac{\cos\theta\cos\phi}{r}$

$\theta_y=\frac{zy}{r^2(x^2+y^2)^{1/2}}=\frac{\cos\theta\sin\phi}{r} \qquad \theta_z=\frac{\partial}{\partial z}\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)=\frac{1}{\frac{x^2+y^2}{z^2}+1}\cdot\sqrt{x^2+y^2}\left(\frac{-1}{z^2}\right)=\frac{-[x^2+y^2]^{1/2}}{r^2}=\frac{-\sin\theta}{r}$

$\phi_x=\frac{\partial}{\partial x}\tan^{-1}(y/x)=\frac{1}{y^2/x^2+1}\cdot\frac{-y}{x^2}=\frac{-y}{y^2+x^2}=\frac{-\sin\phi}{r\sin\theta} \qquad \phi_y=\frac{1}{\frac{y^2}{x^2}+1}\cdot\frac{1}{x}=\frac{x}{y^2+x^2}=\frac{\cos\phi}{r\sin\theta} \qquad \phi_z=0$

$r_{xx}=\frac{\partial}{\partial x}\frac{x}{r}=\frac{r-x[x^2+y^2+z^2]^{-1/2}\cdot\frac{1}{2}2x}{r^2}=\frac{y^2+z^2}{r^3}=\frac{\sin^2\theta\sin^2\phi+\cos^2\theta}{r} \qquad r_{yy}=\frac{x^2+z^2}{r^3}=\frac{\sin^2\theta\cos^2\phi+\cos^2\theta}{r}$

$r_{zz}=\frac{x^2+y^2}{r^3}=\frac{\sin^2\theta}{r}$

$\theta_{xx}=\frac{\partial}{\partial x}\frac{zx}{(x^2+y^2+z^2)(x^2+y^2)^{1/2}}=\frac{zr^2(x^2+y^2)^{1/2}-zx\left[2x(x^2+y^2)^{1/2}+r^2\frac{x}{(x^2+y^2)^{1/2}}\right]}{r^4(x^2+y^2)}=\frac{zr^2y^2-2zx^2(x^2+y^2)}{r^4(x^2+y^2)^{3/2}}=\frac{\cos\theta}{r^2\sin\theta}(\sin^2\phi-2\cos^2\phi\sin^2\theta)$

$\theta_{yy}=\frac{zr^2x^2-2zy^2(x^2+y^2)}{r^4(x^2+y^2)^{3/2}}=\frac{\cos\theta}{r^2\sin\theta}(\cos^2\phi-2\sin^2\phi\sin^2\theta)$

$\theta_{zz}=\frac{\partial}{\partial z}\frac{-[x^2+y^2]^{1/2}}{(x^2+y^2+z^2)}=\frac{[x^2+y^2]^{1/2}\cdot 2z}{(x^2+y^2+z^2)^2}=\frac{2z(x^2+y^2)^{1/2}}{r^4}=\frac{\cos\theta}{r^2\sin\theta}2\sin^2\theta$

$\phi_{xx}=\frac{\partial}{\partial x}\frac{-y}{y^2+x^2}=\frac{y}{(x^2+y^2)^2}\cdot 2x=\frac{2\cos\phi\sin\phi}{r^2\sin^2\theta} \qquad \phi_{yy}=\frac{\partial}{\partial y}\frac{x}{y^2+x^2}=\frac{-x}{(y^2+x^2)^2}\cdot 2y=\frac{-2\cos\phi\sin\phi}{r^2\sin^2\theta} \qquad \phi_{zz}=0$

$$\Delta=\left[\sin^2\theta\sin^2\phi+\cos^2\theta+\sin^2\theta\cos^2\phi+\cos^2\theta+\sin^2\theta\right]\frac{\partial_r}{r}+\left[\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta\right]\partial_r^2$$
$$+\left[\sin^2\phi-2\cos^2\phi\sin^2\theta+\cos^2\phi-2\sin^2\phi\sin^2\theta+2\sin^2\theta\right]\frac{\cos\theta}{r^2\sin\theta}\partial_\theta+\left[\cos^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi+\sin^2\theta\right]\frac{\partial_\theta^2}{r^2}$$
$$+\left[2\cos\phi\sin\phi-2\cos\phi\sin\phi\right]\frac{\partial_\phi}{r^2\sin^2\theta}+\left[\sin^2\phi+\cos^2\phi\right]\frac{\partial_\phi^2}{r^2\sin^2\theta}$$

$$\Delta=\frac{2}{r}\partial_r+\partial_r^2+\frac{\cos\theta}{r^2\sin\theta}\partial_\theta+\frac{1}{r^2}\partial_\theta^2+\frac{1}{r^2\sin^2\theta}\partial_\phi^2$$

a. $u_t-k\Delta u=0=T'R\Theta\Phi-k\left[\frac{2}{r}R'T\Theta\Phi+R''T\Theta\Phi+\frac{\cos\theta}{r^2\sin\theta}\Theta'TR\Phi+\frac{1}{r^2}\Theta''TR\Phi+\frac{1}{r^2\sin^2\theta}\Phi''TR\Theta\right]$

$\frac{T'}{T}=k\left[\frac{2}{r}\frac{R'}{R}+\frac{R''}{R}+\frac{\cos\theta}{r^2\sin\theta}\frac{\Theta'}{\Theta}+\frac{1}{r^2}\frac{\Theta''}{\Theta}+\frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi}\right]=\lambda$

$\lambda\neq 0\Rightarrow T=Ae^{\lambda t} \qquad \lambda=0\Rightarrow T=A$

$2r\frac{R'}{R}+r^2\frac{R''}{R}+\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}+\frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}=\frac{r^2\lambda}{k}$

$r^3\frac{R''}{R}+2r\frac{R'}{R}-\frac{r^2}{k}\lambda=\frac{-\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}-\frac{\Theta''}{\Theta}-\frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}=\delta$

$r^2\frac{R''}{R}+2r\frac{R'}{R}-\frac{r^2}{k}\lambda=\delta$ is too hard to solve

$\pm\cos\theta\sin\theta\frac{\Theta'}{\Theta}+\sin^2\theta\frac{\Theta''}{\Theta}+\sin^2\theta\,\delta=\frac{-\Phi''}{\Phi}=p=m^2$

$\frac{\Phi''}{\Phi}=-p\Rightarrow\Phi=A\cos(m\phi)+B\sin(m\phi)\ m\geq 1 \qquad m=0\Rightarrow\Phi=A \quad (p>0$

$\cos\theta\sin\theta\frac{\Theta'}{\Theta}+\sin^2\frac{\Theta''}{\Theta}+\sin^2\theta\,\delta=p$ is also too nasty to solve

b) $u_{tt} - c^2 \Delta u = 0 = T''R\Theta\phi - c^2\left[\frac{2}{r}R'T\Theta\phi + R''T\Theta\phi + \frac{\cos\theta}{r^2\sin\theta}\Theta'TR\phi + \frac{1}{r^2}\Theta''TR\phi + \frac{1}{r^2\sin^2\theta}\phi''TR\Theta\right]$

$\frac{T''}{T} = c^2\left[\frac{2}{r}\frac{R'}{R} + \frac{R''}{R} + \frac{\cos\theta}{r^2\sin\theta}\frac{\Theta'}{\Theta} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{1}{r^2\sin^2\theta}\frac{\phi''}{\phi}\right] = \lambda$

$T'' = \lambda T \quad \lambda \neq 0 \Rightarrow T = Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t} \qquad \lambda = 0 \Rightarrow T = A + Bt$

The next part is the same as in part a but

$k = c^2$

$r^2\frac{R''}{R} + 2r\frac{R'}{R} - \frac{r^2}{c^2}\lambda = \delta$

$-\frac{\phi''}{\phi} = \rho \Rightarrow \phi = E\cos(m\phi) + B\sin(m\phi) \quad m \geq 1 \qquad m = 0 \Rightarrow \phi = C$

$\cos\theta\sin\theta\frac{\Theta'}{\Theta} + \sin^2\frac{\Theta''}{\Theta} + \sin^2\theta\,\delta = \rho$

Class Exercise 3:

$\ell^2 = \{\{x_k\}_{k\in\mathbb{N}} \text{ s.t } \sum x_k^2 < \infty, \ \langle\{x\},\{y\}\rangle = \sum_{k=1}^{\infty} x_k y_k\}$

Let $\vec{x}_n$ be a converging seq in ℓ^2

$\forall \varepsilon > 0 \ \exists N$ s.t. $\|\vec{x}_n - \vec{x}_m\| < \varepsilon$ for $n, m \geq N$.

$\|\vec{x}_n - \vec{x}_m\|^2 = \langle x_n - x_m, x_n - x_m\rangle = \sum_{k=1}^{\infty}(x_{nk} - x_{mk})^2 < \varepsilon^2$

this must be true for each k because $(x_{nk} - x_{mk})^2 \geq 0 \ \forall k$.

$\Rightarrow (x_{nk} - x_{mk})^2 < \varepsilon^2 \ \forall k \Rightarrow |x_{nk} - x_{mk}| < \varepsilon \qquad x_{nk}, x_{mk} \in \mathbb{R}^n$

and $\mathbb{R}^n$ is complete so $x_{nk} \xrightarrow{m\to\infty} \tilde{x}_k$

Now show that $\tilde{x} \in \ell^2$, $\sum_{k=1}^{\infty}(\tilde{x}_k)^2 < \infty$

Lemma 1: Because x_n is cauchy it must be bounded.

$\|x_n\| \leq C$ for some $C \in \mathbb{R}$

Choose $\varepsilon = 1$ then $\exists N_1$ s.t.

$\|x_n - x_m\| < 1$ for $n, m \geq N$

fix $m > N_1$ then $\|x_n\| \leq \|x_m\| + 1 =: C$

Show $\sum_{k=1}^{N}(\tilde{x}_k)^2 \leq C^2 \ \forall N$

$x_{n_k} \xrightarrow{n\to\infty} \tilde{x}_k \Rightarrow \sum_{k=1}^{N}(x_{nk})^2 \xrightarrow{n\to\infty} \sum_{k=1}^{N}(\tilde{x}_k)^2 \leq$

$\sum_{k=1}^{N}(x_{nk})^2 \leq C^2$ for large enough $n \Rightarrow \sum_{k=1}^{N}(\tilde{x}_k)^2 \leq C^2 \Rightarrow \lim_{N\to\infty}\sum_{k=1}^{N}(x_k^2) = \sum_{k=1}^{\infty}(\tilde{x}_k)^2 \leq C^2$ □

Now Show $\vec{x}_n \xrightarrow{n\to\infty} \tilde{x}$ in ℓ^2 norm

Lemma 2: $\|\vec{x}_n - \tilde{x}\|^2 \leq \overline{\lim_{m\to\infty}}\|\vec{x}_n - \vec{x}_m\|^2 \leq \varepsilon^2$

$\|\vec{x}_n - \tilde{x}\|^2 = \sum_{k=1}^{\infty}|x_{nk} - \tilde{x}_k|^2 = \lim_{K\to\infty}\sum_{k=1}^{K}|x_{nk} - \tilde{x}_k|^2 = \lim_{K\to\infty}\left(\lim_{m\to\infty}\sum_{k=1}^{K}|x_{nk} - x_{mk}|^2\right)$

$\leq \lim_{K\to\infty}\left(\lim_{m\to\infty}\|x_n - x_m\|^2\right) = \lim_{m\to\infty}\|x_n - x_m\|^2$

Now show $\vec{X}_n \xrightarrow{n\to\infty} \tilde{x}$

$\|\vec{X}_n - \tilde{x}\| \xrightarrow{n\to\infty} 0$

$\|\vec{X}_n - \tilde{x}\|^2 = \sum_{k=1}^{\infty} |x_{nk} - x_k|^2$

Let $\varepsilon > 0$ $\exists N$ s.t. $\|x_n - x_m\|^2 < \varepsilon^2$ when $n, m \geq N$

$\Rightarrow \overline{\lim_{m\to\infty}} \|x_n - x_m\|^2 \leq \varepsilon$

$\geq \|x_n - \tilde{x}\|^2 \Rightarrow \|x_n - \tilde{x}\| \leq \varepsilon$ ∎

Class Exercise 4:

(0) → (1)

$H = \overline{\text{span}}\{u_k\} \Rightarrow x = \sum_{j=1}^{\infty} c_j u_j$, where only a finite colection of the $c_j \neq 0$.

$\langle x, u_k \rangle = \langle \sum_{j=1}^{\infty} c_j u_j, u_k \rangle = \sum_{j=1}^{\infty} \langle c_j u_j, u_k \rangle = c_k$

$x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k$

(1) → (2)

Lemma 1: $\{z_k\} \to z \Rightarrow \langle z_k, y \rangle \to \langle z, y \rangle$

$|\langle z_k, y \rangle - \langle z, y \rangle| = |\langle z_k - z, y \rangle| \leq |z_k - z| |y| \to 0$

if also $\{w_k\} \to w$ then $\langle z_k, w_k \rangle \to \langle z, w \rangle$

$\langle z, w \rangle - \langle z_k, w_k \rangle = \langle z, w \rangle - \langle z_k, w \rangle^{\to 0} + \langle z_k, w \rangle - \langle z_k, w_k \rangle$

$\Rightarrow |\langle z_k, w - w_k \rangle| \leq |z_k| |w - w_k| \leq M |w - w_k| \to 0$ because cauchy seq are bounded.

So $\|z_k\| \to \|z\|$ ∎

$x \leftarrow \sum_k^N \langle x, u_k \rangle u_k$

$\|\sum_{k=1}^N \langle x, u_k \rangle u_k\|^2 \to \|x\|^2$ by the Lemma.

$= \sum_{k,l=1}^N \langle x, u_k \rangle u_k \cdot \langle x, u_l \rangle u_l$ $\quad u_k u_l = \delta_{lk}$

$= \sum_{k=1}^N \langle x, u_k \rangle^2 \longrightarrow \|x\|^2$ ∎

(2) → (3) $T: H \to \ell^2 \quad x \longmapsto \{\langle x, u_k \rangle\}_{k=1}^{\infty}$

a. Show T is linear

$T(x+y) = \{\langle x+y, u_k \rangle\}_{k=1}^{\infty} = \{\langle x, u_k \rangle + \langle y, u_k \rangle\}_{k=1}^{\infty} = T(x) + T(y)$

$T(cx) = \{\langle cx, u_k \rangle\}_{k=1}^{\infty} = \{c\langle x, u_k \rangle\}_{k=1}^{\infty} = cT(x)$

b. T is one-to-one and onto. It is sufficient to show that T has a linear inverse

$T: H \to \ell^2 \quad x \mapsto \{\langle x, u_k\rangle\}_{k=0}^{\infty}$

$T^{-1}: \ell^2 \to H \quad \{c_k\} \mapsto \sum_{k=1}^{\infty} c_k u_k = x$ by (1)

$T^{-1}(a\{c_k\}+b\{d_k\}) = \sum_{k=1}^{\infty}(ac_k+bd_k)u_k = \sum_{k=1}^{\infty} ac_k u_k + \sum_{k=1}^{\infty} bd_k u_k = a\sum_{k=1}^{\infty} c_k u_k + b\sum_{k=1}^{\infty} d_k u_k$

$= aT^{-1}(c_k) + bT^{-1}(d_k)$ ▨

c. T preserves norm.

by (2) $\sum_{n=1}^{\infty}\langle x, u_k\rangle^2 = \|x\|^2$

$\Rightarrow |Tx|_{\ell^2}^2 = |x|_H^2$

(3)→(0)

$T: H \to \ell^2 \qquad T^{-1}: \ell^2 \to H$

$x \mapsto \{\langle x, u_k\rangle\} \qquad \{c_k\} \in \ell^2 \mapsto \sum_{k=1}^{\infty} c_k u_k$

every x can be written in the form $\lim_{N\to\infty}\sum_{k=1}^{N} c_k u_k$. Every sum $\sum_{k=1}^{N} c_k u_k \in \operatorname{span}\{u_k\}$

∴ $\forall x \in \overline{\operatorname{span}\{u_k\}}$ because all xs can be written as a limit point of the span.

Class Exercise 5: $T\vec{x} = A\vec{x} \quad \|T\|_{op} = \sup_{\|x\|=1} |Tx|$

a. $x = \sum_{n=1}^{\infty} c_j u_j$ where u_j is the O.N. basis for A. $\|x\|^2 = \sum_{j=1}^{\infty} c_j^2 = 1$

$|Tx|^2 = |Ax|^2 = |A\sum_{j=1}^{\infty} c_j u_j|^2 = |\sum_{j=1}^{\infty} c_j A u_j|^2 = |\sum_{j=1}^{\infty} c_j \lambda_j u_j|^2 =$

$= \sum_{j=1}^{\infty} c_j^2\lambda_j^2 \leq \sum_{j=1}^{\infty} c_j^2 \cdot \lambda_m^2$ where $\lambda_m = \max_i \{\lambda_i\}$

$= \lambda_m^2 \sum_{j=1}^{\infty} c_j^2 = \lambda_m^2 \Rightarrow |Tx|^2 \leq \lambda_m^2$

if $x = \sum_{j=1}^{\infty} c_j u_j$ and $c_j = \begin{cases} 0 & j \neq m \\ 1 & j = m\end{cases}$ then $|Tx|^2 = \lambda_m^2$

so $\|Tx\|_{op} = \lambda_m$ ▨

b. $|Tx|^2 = (Ax)\cdot(Ax) = (Ax)^T(Ax) = x^T(A^TAx) = (A^TAx)\cdot x \quad \operatorname{span}\{\bar{u}_i\}$ = O.N. basis for A^TA

$= (A^TA\sum_{i=1}^{\infty}\bar{c}_i\bar{u}_i)\cdot\sum_{j=1}^{\infty}\bar{c}_j\bar{u}_j = (\sum_{i=1}^{\infty}\bar{c}_i\bar{\lambda}_i\bar{u}_i)\cdot(\sum_{j=1}^{\infty}\bar{c}_j\bar{u}_j) \quad \bar{u}_i\cdot\bar{u}_j = \delta_{ij}$

$= \sum_{i=1}^{\infty}\bar{c}_i^2\bar{\lambda}_i \leq \bar{\lambda}_m\sum_{i=1}^{\infty}\bar{c}_i^2 = \bar{\lambda}_m$

$\sum_{j=1}^{\infty}\bar{c}_i^2\bar{\lambda}_i = \bar{\lambda}_m$ if $c_i = \begin{cases} 0 & i \neq m \\ 1 & i = m\end{cases}$ so $\|Tx\| = \sqrt{\bar{\lambda}_m}$

c. $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{bmatrix}1&2\\0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \quad A^TA = \begin{bmatrix}1&0\\2&3\end{bmatrix}\begin{bmatrix}1&2\\0&3\end{bmatrix} = \begin{bmatrix}1&2\\2&13\end{bmatrix} \quad \begin{vmatrix}1-\lambda&2\\2&13-\lambda\end{vmatrix} = 0 \Rightarrow (1-\lambda)(13-\lambda)-4=0$

$13-14\lambda+\lambda^2-4=0 \Rightarrow \lambda^2-14\lambda+9=0 \quad \lambda = \frac{14 \pm \sqrt{196-36}}{2} = 7 \pm \frac{\sqrt{160}}{2} = 7 \pm 2\sqrt{10}$

$\boxed{\|T\|_{op}^2 = 7+2\sqrt{10}}$ ← (so $\|T\|_{op} = \sqrt{\%} > 3 = \max|\lambda|$, λ = eval of $\begin{bmatrix}1&2\\0&3\end{bmatrix}$)