Section 7.1

(1) Derive the three-dimensional maximum principle from the MVP.

If Ω is a bounded, open connected subset of $\mathbb{R}^n$ and if

$$u \in C^2(\Omega) \cap C(\bar{\Omega}), \quad \Delta u = 0$$

then the max of u occurs on $\partial\Omega$. If it occurs at any (interior) point $x_0 \in \Omega$, then u is constant.

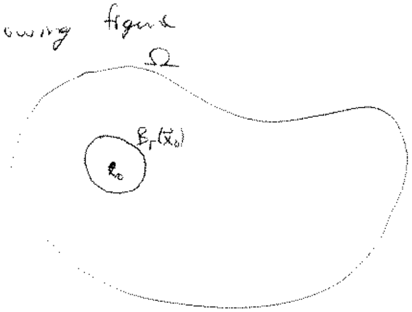
Proof: We follow a proof given in Strauss, section 6.3.

Since $u \in C(\bar{\Omega})$, u must attain its maximum and minimum value somewhere in $\bar{\Omega}$, since $\bar{\Omega}$ is a compact set. Our goal is to show the second half of the statement, namely

If the $\max_{\vec{x} \in \bar{\Omega}} u(\vec{x})$ occurs at some interior point $x_0 \in \Omega$, then u is constant.

Suppose that $\max_{\vec{x} \in \bar{\Omega}} u(\vec{x})$ occurs at $\vec{x}_0 \in \Omega^o (= \Omega)$. Let $M = \max_{\vec{x} \in \bar{\Omega}} u(\vec{x})$.

Consider the following figure



We surround $\vec{x}_0$ with a ball, say $B_r(\vec{x}_0)$, such that $B_r(\vec{x}_0) \subset \Omega$.

By the MVP for balls (class exercise 7), we have

$$M = u(\vec{x}_0) = \fint_{B_r^n(\vec{x}_0)} u(\vec{x})\, dV_n \le \fint_{B_r^n(\vec{x}_0)} M\, dV_n = M$$

This implies that $u(\vec{x}) = M$ everywhere in $B_r(\vec{x}_0)$. To see this, suppose $\exists\, \vec{x}_1 \in B_r(\vec{x}_0)$ such that $u(\vec{x}_1) < M$. Since u is continuous, there is a $\delta > 0$ such that $u(\vec{x}) < M$ for all $\vec{x} \in B_\delta(\vec{x}_1)$

Then

$$\fint_{B_r(\vec{x}_0)} u(\vec{x})\, dV_n = \frac{1}{|B_1^n| r^n} \int_{B_r(\vec{x}_0)} u(\vec{x})\, dV_n$$

$$= \frac{1}{|B_1| r^n} \left[\int_{B_r(\vec{x}_0) \setminus B_\delta(\vec{x}_1)} u(\vec{x})\, dV_n + \int_{B_\delta(\vec{x}_1)} u(\vec{x})\, dV_n \right]$$

$$= \frac{1}{|B_1| r^n} \left[\int_{B_r(\vec{x}_0) \setminus B_\delta(\vec{x}_1)} M\, dV_n + \int_{B_\delta(\vec{x}_1)} u(\vec{x})\, dV_n \right]$$

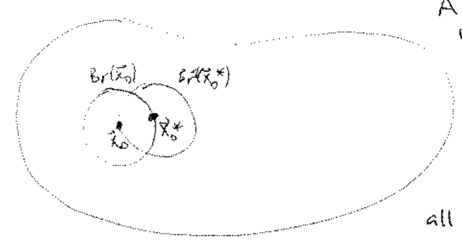
$$\text{strict} \longrightarrow < \frac{1}{|B_1| r^n} \left[\int_{B_r(\vec{x}_0) \setminus B_\delta(\vec{x}_1)} M + \int_{B_\delta(\vec{x}_1)} M \right]$$

$$= \frac{1}{|B_1| r^n} M \left[|B_r(\vec{x}_0) \setminus B_\delta(\vec{x}_1)| + |B_\delta(\vec{x}_1)| \right]$$

$$= \frac{1}{|B_1| r^n} M\, |B_r(\vec{x}_0)| = M$$

which is a contradiction, since $\fint_{B_r(\vec{x}_0)} u(\vec{x})\, dV_n = M$.

Figure 2 shows how we can continue this process:



A more correct argument is to note that you've shown $\{x \in \Omega \text{ s.t. } u(x) = u(x_0)\}$ is open. It's automatically closed. Ω connected $\Rightarrow$ it is all of Ω. ~~So~~ So $u \equiv u(x_0)$ in Ω and also in $\bar{\Omega}$ by continuity

That is, we choose $\vec{x}_0^*$ on the boundary of $B_r(\vec{x}_0)$. Then $u(\vec{x}_0^*) = M$ and, by the same argument as above, $u(\vec{x}) = M$ for all $\vec{x} \in B_{r^*}(\vec{x}_0^*)$. We choose r^* small enough so $B_{r^*}(\vec{x}_0^*) \subset \overline{\Omega}$. We continue the process until we cover $\overline{\Omega}$. This shows that if $M = \max_{\vec{x} \in \overline{\Omega}} u(\vec{x})$ occurs at some point $\vec{x}_0 \in \Omega$, then $u(\vec{x}) = M$ for all $\vec{x} \in \overline{\Omega}$, i.e., u is constant in $\overline{\Omega}$.

By the contrapositive of the last sentence, we have that if u is nonconstant in $\overline{\Omega}$, then the maximum of u does not occur at any interior point $x_0 \in \Omega$. However, since u is continuous and $\overline{\Omega}$ is compact, u must attain its maximum somewhere in $\overline{\Omega}$. Since it cannot be an interior point of Ω, the maximum of u must occur on the boundary. $\square$

(3) Prove the uniqueness of the Robin problem $\frac{\partial u}{\partial n} + a(\vec{x})u(\vec{x}) = 0$ provided that $a(\vec{x}) > 0$ on the boundary.

Soln: We wish to show that the problem

$$\text{RP}\begin{cases} -\Delta u = F & \text{in } \Omega \\ \frac{\partial u}{\partial n} + a(\vec{x})u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution. We begin by defining $w := u - v$, where u and v both solve RP. We recall the identity that

$$\nabla \cdot (w\nabla w) = \nabla w \cdot \nabla w + w\Delta w = |\nabla w|^2 + w\Delta w$$

Integrating both sides of the above equality gives

$$(*) \quad \underset{①}{\int_{\Omega} \nabla \cdot (w\nabla w)\, dV_n} = \underset{②}{\int_{\Omega} |\nabla w|^2\, dV_n} + \underset{③}{\int_{\Omega} w\Delta w\, dV_n}.$$

① If we apply the Divergence theorem to ①, we obtain

$$\int_{\Omega} \nabla \cdot (w\nabla w)\, dV_n = \int_{\partial\Omega} (w\nabla w)\cdot \hat{n}\, dV_{n-1} = \int_{\partial\Omega} w\frac{\partial w}{\partial n}\, dV_{n-1}.$$

We also note that w solves

$$\text{RP}_w\begin{cases} \Delta w = 0 \\ \frac{\partial w}{\partial n} + a(\vec{x})w = 0, \end{cases}$$

Since $\Delta w = \Delta(u-v) = \Delta u - \Delta v = -F + F = 0$ in Ω

$$\frac{\partial w}{\partial n} + aw = \frac{\partial(u-v)}{\partial n} + a(u-v) = \frac{\partial u}{\partial n} + au - \frac{\partial v}{\partial n} - av = 0 - 0 = 0 \text{ on } \partial\Omega$$

③ Thus ③ becomes $\int_{\Omega} w \underbrace{\Delta w}_{0}\, dV_n = 0$

$\Rightarrow$ $(*)$ becomes $\int_{\partial\Omega} w\frac{\partial w}{\partial n}\, dV_{n-1} = \int_{\Omega} |\nabla w|^2\, dV_n$

on $\partial\Omega$, $\frac{\partial w}{\partial n} = -aw$, so the above equation is equivalent to

$$\int_{\partial\Omega} w(-aw)\, dV_{n-1} = \int_{\Omega} |\nabla w|^2\, dV_n$$

$$\Leftrightarrow \quad \underset{④}{\int_{\partial\Omega} -aw^2\, dV_{n-1}} = \underset{⑤}{\int_{\Omega} |\nabla w|^2\, dV_n}$$

Since $a > 0$ on $\partial\Omega$, ④ ≤ 0. Also, ⑤ ≥ 0. This implies that $\int_{\partial\Omega} -aw^2\, dV_{n-1} = \int_{\Omega} |\nabla w|^2 dV_n = 0$.

Since $-aw^2$ is non-positive, and since $|\nabla w|^2$ is non-negative, $-aw^2 = 0$ on $\partial\Omega$ and $\nabla w = 0$ in Ω

Thus $\nabla w = 0 \Rightarrow w$ is constant throughout Ω.

Since $a > 0$ on $\partial\Omega$, and since $-aw^2 = 0$ on $\partial\Omega$, $w = 0$ on $\partial\Omega$

Thus $w \equiv 0$ in $\overline{\Omega}$. This implies that $u \equiv v$ in $\overline{\Omega}$, and, hence that solutions to the Robin problem are unique. □

(5) Prove Dirichlet's Principle for the Neumann boundary condition.

It asserts that among <u>all</u> real-valued functions $w(\vec{x})$ on D the quantity $E[w] = \frac{1}{2}\iiint_D |\nabla w|^2\, d\vec{x} - \iint_{\partial D} hw\, dS$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$\text{NP}\begin{cases} -\Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial n} = h(\vec{x}) & \text{on } \partial D \end{cases}$$

It is required to assume that the average of the given function $h(\vec{x})$ is zero.

Notice three features of this principle

(i) There is no constant at all on the trial functions $w(\vec{x})$

(ii) The function $h(\vec{x})$ appears in the energy

(iii) The functional $E[w]$ does not change if a constant is added to w.

Proof: Let $u(\vec{x})$ be a (not necessarily unique) harmonic function that satisfies (NP) following the method in Section 7.1, we define $v := w - u$, so $w = u + v$. Then

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 d\vec{x} - \iint_{\partial D} h w \, dS$$

$$= \frac{1}{2}\iiint_D |\nabla(u+v)|^2 d\vec{x} - \iint_{\partial D} h(u+v)\, dS$$

$$= \frac{1}{2}\iiint_D \left[|\nabla u|^2 + 2\nabla u\cdot\nabla v + |\nabla v|^2\right] d\vec{x} - \iint_{\partial D} h u \, dS - \iint_{\partial D} h v \, dS$$

$$(**)\quad E[w] = \underbrace{\frac{1}{2}\iiint_D |\nabla u|^2 d\vec{x} - \iint_{\partial D} h u \, dS}_{E[u]} + \frac{1}{2}\iiint_D |\nabla v|^2 d\vec{x} - \iint_{\partial D} h v \, dS + \iiint_D \nabla u\cdot\nabla v.$$

Now, $\iint_{\partial D} h v \, dS = \iint_{\partial D} \frac{\partial u}{\partial n} v \, dS$, since u satisfies (NP).

Since $\nabla\cdot(v\nabla u) = \nabla v\cdot\nabla u + v\Delta u$, we have, after integrating both sides over D and applying the Divergence Theorem on the left side, that

$$(***)\quad \iint_{\partial D} v \nabla u\cdot n \, dS = \iiint_D \nabla v\cdot\nabla u \, d\vec{x} + \iiint_D v \underbrace{\Delta u}_{=0} d\vec{x}$$

$$\Leftrightarrow \iint_{\partial D} v \frac{\partial u}{\partial n} dS = \iiint_D \nabla v\cdot\nabla u \, d\vec{x}$$

$$\Leftrightarrow \iint_{\partial D} h v \, dS = \iiint_D \nabla v\cdot\nabla u \, d\vec{x}$$

Then (**) becomes

$$E[w] = E[u] + \frac{1}{2}\iiint_D |\nabla v|^2 d\vec{x} - \iiint_D \nabla v\cdot\nabla u \, d\vec{x} + \iiint_D \nabla v\cdot\nabla u \, d\vec{x}$$

$$= E[u] + \frac{1}{2}\iiint_D |\nabla v|^2 d\vec{x}.$$

Since $|\nabla v|^2 \geq 0$, $\frac{1}{2}\iiint_D |\nabla v|^2 d\vec{x} \geq 0$.

Hence $E[w] \geq E[u]$, so the energy is smallest when $w = u$.

Note: The assumption that the average of $h(\vec{x})$ is zero is a consequence of the following. by (***), we have

$$\iint_{\partial D} g \frac{\partial u}{\partial n} dS = \iiint_D \nabla g\cdot\nabla u \, d\vec{x} + \iiint_D g \Delta u \, d\vec{x}$$

If we let $g \equiv 1$, this becomes

$$\iint_{\partial D} h \, dS = \iint_{\partial D} \frac{\partial u}{\partial n} dS = \iiint_D \underbrace{\Delta u}_{0} d\vec{x} = 0$$

Thus $\frac{1}{4\pi R^2}\iint_{\partial D} h \, dS = 0$ for any $R > 0$, so the average value of h must be zero if u is to solve (NP). □

Section 7.4:

(6) a) Find the Green's function for the half-plane $D = \{(x,y) : y > 0\}$

b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$.

c) Calculate the solution with $u(x,0) = 1$.

Sol: a) <u>Claim</u>: The Green's function is given by

$$G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0| - \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0^*|$$

where $\vec{x}=(x,y)$, $\vec{x}_0=(x_0,y_0)$ and $\vec{x}_0^*$ is the reflection of $\vec{x}_0$ about the x-axis, namely $\vec{x}_0^* = (x_0,-y_0)$.

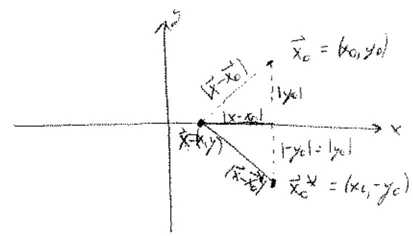


Figure 1

We need to show that G satisfies the following 3 properties

(i) G possesses continuous second derivatives with respect to $\vec{x}$ and $\Delta G = 0$ in D, except at $\vec{x}=\vec{x}_0$.

(ii) $G(\vec{x}) = 0$ for $\vec{x}\in\partial D$.

(iii) The function $G(\vec{x},\vec{x}_0) - \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0|$ is finite at $\vec{x}_0$ and has continuous second derivatives everywhere and is harmonic at $\vec{x}_0$.

Pf: (i) $G(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\cdot 2\ln|\vec{x}-\vec{x}_0| - \frac{1}{4\pi}\cdot 2\ln|\vec{x}-\vec{x}_0^*|$

$$= \frac{1}{4\pi}\left(\ln|\vec{x}-\vec{x}_0|^2 - \ln|\vec{x}-\vec{x}_0^*|^2\right)$$

$$= \frac{1}{4\pi}\left[\ln|(x,y)-(x_0,y_0)|^2 - \ln|(x,y)-(x_0,-y_0)|^2\right]$$

$$= \frac{1}{4\pi}\left\{\ln\left[(x-x_0)^2+(y-y_0)^2\right] - \ln\left[(x-x_0)^2+(y+y_0)^2\right]\right\}$$

$$\frac{\partial G}{\partial x}(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\left[\frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2}\right]$$

$$\frac{\partial^2 G}{\partial x^2}(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\left[\frac{\left[(x-x_0)^2+(y-y_0)^2\right]2 - 2(x-x_0)\,2(x-x_0)}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} - \frac{\left[(x-x_0)^2+(y+y_0)^2\right]2 - 2(x-x_0)\,2(x-x_0)}{\left[(x-x_0)^2+(y+y_0)^2\right]^2}\right]$$

$$\frac{\partial^2 G}{\partial x^2} = \frac{1}{4\pi}\left[\frac{2|\vec{x}-\vec{x}_0|^2 - 4(x-x_0)^2}{|\vec{x}-\vec{x}_0|^4} - \frac{2|\vec{x}-\vec{x}_0^*|^2 - 4(x-x_0)^2}{|\vec{x}-\vec{x}_0^*|^4}\right]$$

similarly, $$\frac{\partial^2 G}{\partial y^2} = \frac{1}{4\pi}\left[\frac{2|\vec{x}-\vec{x}_0|^2 - 4(y-y_0)^2}{|\vec{x}-\vec{x}_0|^4} - \frac{2|\vec{x}-\vec{x}_0^*|^2 - 4(y+y_0)^2}{|\vec{x}-\vec{x}_0^*|^4}\right]$$

We note that $\frac{\partial^2 G}{\partial x^2}$ and $\frac{\partial^2 G}{\partial y^2}$ are continuous everywhere in D except $\vec{x}=\vec{x}_0$.

Also, $$\Delta G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \frac{1}{4\pi}\left[\frac{4|\vec{x}-\vec{x}_0|^2 - 4\left[(x-x_0)^2+(y-y_0)^2\right]}{|\vec{x}-\vec{x}_0|^4} - \frac{4|\vec{x}-\vec{x}_0^*|^2 - 4\left[(x-x_0)^2+(y+y_0)^2\right]}{|\vec{x}-\vec{x}_0^*|^4}\right]$$

$$= \frac{1}{4\pi}\left[\frac{4|\vec{x}-\vec{x}_0|^2 - 4|\vec{x}-\vec{x}_0|^2}{|\vec{x}-\vec{x}_0|^4} - \frac{4|\vec{x}-\vec{x}_0^*|^2 - 4|\vec{x}-\vec{x}_0^*|^2}{|\vec{x}-\vec{x}_0^*|^4}\right]$$

$= 0.$ (you've shown this before)

As can be seen in figure 1 from geometry, $|\vec{x}-\vec{x}_0| = |\vec{x}-\vec{x}_0^*|$; by the Pythagorean Theorem,
$$|\vec{x}-\vec{x}_0|^2 = |x-x_0|^2 + |y_0|^2$$
$$|\vec{x}-\vec{x}_0^*| = |x-x_0|^2 + |-y_0|^2 = |x-x_0|^2 + |y_0|^2 = |\vec{x}-\vec{x}_0|$$

(ii) Since $|\vec{x}-\vec{x}_0| = |\vec{x}-\vec{x}_0^*|$,
$G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\left[\ln|\vec{x}-\vec{x}_0| - \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0^*|\right] = 0$ for $\vec{x}\in\partial D = \{(x,0): x\in$

(iii) $G(\vec{x},\vec{x}_0) - \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0| = -\frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0^*|$. This function is finite at $\vec{x}$ and has continuous second derivatives everywhere and is harmonic at $\vec{x}_0$

Therefore $\boxed{G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0| - \frac{1}{2\pi}\ln|x-x_0^*|}$

b) By theorem 2 (p 181 in Strauss' text)

$$u(\vec{x}_0) = \int_{\partial D} h(\vec{x}) \frac{\partial G(\vec{x},\vec{x}_0)}{\partial n} \, dS + \iint_D f(\vec{x}) G(\vec{x},\vec{x}_0),$$

where u solves $\begin{cases} \Delta u = f & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$

In our case, we want $f = 0$. Thus

$$u(\vec{x}_0) = \int_{\partial D} h(\vec{x}) \frac{\partial G(\vec{x},\vec{x}_0)}{\partial r} \, dS, \quad \text{where } \partial D = \{(x,0) : x \in \mathbb{R}\}$$

The outward normal for D is $\hat{n} = -\hat{j}$, so $\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial y}$.

Now, $G(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\left[\ln|\vec{x}-\vec{x}_0|^2 - \ln|\vec{x}-\vec{x}_0^*|^2\right]$

$$\Rightarrow G(x,y,x_0,y_0) = \frac{1}{4\pi}\left\{\ln\left[(x-x_0)^2+(y-y_0)^2\right] - \ln\left[(x-x_0)^2+(y+y_0)^2\right]\right\}$$

$$\Rightarrow \frac{\partial G}{\partial y}(x,y,x_0,y_0) = \frac{1}{4\pi}\cdot\left(\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2}\right)$$

on ∂D, we have

$$\frac{\partial G}{\partial y}(x,0,x_0,y_0) = \frac{1}{2\pi}\cdot\frac{-y_0 - y_0}{(x-x_0)^2+(y_0)^2} = -\frac{1}{\pi}\frac{y_0}{(x-x_0)^2+(y_0)^2}$$

$$\Rightarrow \boxed{u(x_0,y_0) = \int_{-\infty}^{\infty} h(x,0)\,\frac{1}{\pi}\frac{y_0}{(x-x_0)^2+(y_0)^2}\,dx}$$

$\therefore$ When $h(x,0) = 1$, we have

$$u(x_0,y_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y_0}{(x-x_0)^2+(y_0)^2}\,dx = \frac{1}{\pi}\lim_{b\to\infty}\int_{-b}^{b}\frac{y_0}{(x-x_0)^2+y_0^2}\,dx$$

$$\frac{1}{\pi}\lim_{b\to\infty} y_0\cdot\frac{1}{y_0}\arctan\left(\frac{x-x_0}{y_0}\right)\Big|_{x=-b}^{b}$$

$$= \frac{1}{\pi}\lim_{b\to\infty}\left[\arctan\left(\frac{b-x_0}{y_0}\right) - \arctan\left(\frac{-b-x_0}{y_0}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{\pi}(\pi) = 1.$$

$$\Rightarrow \boxed{u(x_0,y_0) = 1}$$

(11) a) Verify that $G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\log\rho - \frac{1}{2\pi}\log\left(\frac{a}{r_0}\rho^*\right)$, where $r_0 = |\vec{x}_0|$, $\rho = |\vec{x}-\vec{x}_0|$, and $\rho^* = |\vec{x}-\vec{x}_0^*|$, is the Green's function for the disk D.

b) Use it to recover the Poisson formula.

Soln: We need to show that G satisfies (i)–(iii) from problem 6

(i) $G(\vec{x},\vec{x}_0) = \frac{1}{4\pi}\left[\log|\vec{x}-\vec{x}_0|^2 - \log\left(\frac{a}{r_0}|\vec{x}-\vec{x}_0^*|^2\right)\right]$. If $\vec{x} = (x,y)$ and $\vec{x}_0 = (x_0$

$$G(x,y,x_0,y_0) = \frac{1}{4\pi}\left\{\log\left((x-x_0)^2+(y-y_0)^2\right) - \log\frac{a}{r_0}\left[(x-x_0^*)^2+(y-y_0^*)^2\right]\right\}$$

as in (6)

$$\Rightarrow \frac{\partial^2 G}{\partial x^2}(x,y,x_0,y_0) = \frac{1}{4\pi}\left\{\frac{\left[(x-x_0)^2+(y-y_0)^2\right]2 - 4(x-x_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} - \frac{\left[(x-x_0^*)^2+(y-y_0^*)^2\right]2 - 4(x}{\left[(x-x_0^*)^2+(y-y_0^*)^2\right]^2}\right.$$

and

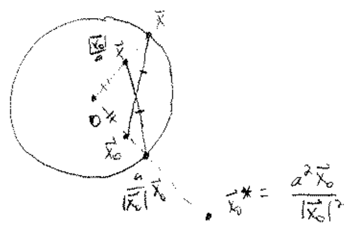
$$\frac{\partial^2 G}{\partial y^2}(x,y,x_0,y_0) = \frac{1}{4\pi}\left\{\frac{\left[(x-x_0)^2+(y-y_0)^2\right]2 - 4(y-y_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} - \frac{\left[(x-x_0^*)^2+(y-y_0^*)^2\right]2 - 4(y-y_0^*)^2}{\left[(x-x_0^*)^2+(y-y_0^*)^2\right]^2}\right.$$

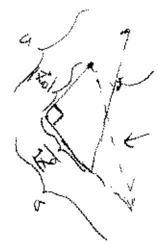
Now, $\frac{\partial^2 G}{\partial x^2}$ and $\frac{\partial^2 G}{\partial y^2}$ are continuous everywhere except at $\vec{x} = \vec{x}_0 = (x_0,y_0)$

Also, $\Delta G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \frac{1}{4\pi}\left[\underbrace{\frac{4\rho^2 - 4\rho^2}{\rho^4}}_{0} - \underbrace{\frac{4(\rho^*)^2 - 4(\rho^*)^2}{(\rho^*)^4}}_{\substack{0 \\ \text{(needed for part (iii))}}}\right] = 0.$

(ii) On ∂D, $|\vec{x}| = a$.

(13)





$$|\vec{x}-\vec{x}_0| = \rho$$

$$\leftarrow \left|\frac{|\vec{x}_0|}{a}\vec{x} - \frac{a}{|\vec{x}_0|}\vec{x}_0\right|$$

$$= \frac{|\vec{x}_0|}{a}\left|\vec{x} - \frac{a^2}{|\vec{x}_0|^2}\vec{x}_0\right|$$

$$\frac{|\vec{x}_0|}{a}|\vec{x}-\vec{x}_0^*| = \frac{|\vec{x}_0|}{a}\rho^*$$

$\Rightarrow$ If $|\vec{x}| = a$, then $\boxed{\rho = \frac{|\vec{x}_0|}{a}\rho^* = \frac{r_0}{a}\rho^*}$ (1)

Thus, if $|\vec{x}| = a$,

$$G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\log\rho - \frac{1}{2\pi}\log\left(\frac{a}{r_0}\rho^*\right) = \frac{1}{2\pi}\log\rho - \frac{1}{2\pi}\log\left(\frac{a}{r_0}\cdot\frac{r_0}{a}\rho\right) = 0.$$

(iii) $G(\vec{x},\vec{x}_0) - \frac{1}{2\pi}\log\rho = -\frac{1}{2\pi}\log\frac{a}{r_0}\rho^* = -\frac{1}{2\pi}\log\frac{a}{r_0}|\vec{x}-\vec{x}_0^*|$.

This function is finite at $\vec{x}_0$, has continuous second derivative everywhere in D, and is harmonic at $\vec{x}_0$ (see part i)

Therefore $G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\log\rho - \frac{1}{2\pi}\log\left(\frac{a}{r_0}\rho^*\right)$ is the Green's function for the disk of radius a centered at the origin.

(b) By Theorem 2 (p.181), we have

$$u(\vec{x}_0) = \int_{\partial D} h(\vec{x})\,\frac{\partial G(\vec{x},\vec{x}_0)}{\partial n}\,ds$$

Now $\frac{\partial G}{\partial n} = \nabla G\cdot\hat{n}$, where $\hat{n} = \frac{\vec{x}}{a}$ on ∂D.

$$\nabla G = \frac{1}{2\pi}\cdot\frac{1}{\rho}\nabla\rho - \frac{1}{2\pi}\frac{r_0}{a}\cdot\frac{1}{\rho^*}\cdot\frac{a}{r_0}\nabla\rho^* = \frac{1}{2\pi}\frac{\nabla\rho}{\rho} - \frac{1}{2\pi}\frac{\nabla\rho^*}{\rho^*}.$$

Now, since $\rho^2 = |\vec{x}-\vec{x}_0|^2$ we have, after differentiating both sides,

(14)

that $2\rho\nabla\rho = \nabla\left[(x-x_0)^2+(y-y_0)^2\right]$

$$= (2(x-x_0),\ 2(y-y_0))$$

$$= 2(\vec{x}-\vec{x}_0)$$

$$\Rightarrow \nabla\rho = \frac{(\vec{x}-\vec{x}_0)}{\rho}.$$

Similarly, $\nabla\rho^* = \frac{(\vec{x}-\vec{x}_0^*)}{\rho^*}$

$$\Rightarrow \nabla G = \frac{1}{2\pi}\left(\frac{\nabla\rho}{\rho} - \frac{\nabla\rho^*}{\rho^*}\right) = \frac{1}{2\pi}\left[\frac{(\vec{x}-\vec{x}_0)}{\rho^2} - \frac{(\vec{x}-\vec{x}_0^*)}{(\rho^*)^2}\right]$$

Now, $\vec{x}_0^* = \left(\frac{a}{r_0}\right)^2\vec{x}_0$ and $\rho^* = \frac{a}{r_0}\rho$ for $\vec{x}\in\partial D$.

Thus, for $\vec{x}\in\partial D$, $\nabla G = \frac{1}{2\pi}\left\{\frac{\vec{x}-\vec{x}_0}{\rho^2} - \frac{\left[\vec{x} - \left(\frac{a}{r_0}\right)^2\vec{x}_0\right]}{\left(\frac{a}{r_0}\rho\right)^2}\right\}$

$$= \frac{1}{2\pi}\left\{\frac{\vec{x}-\vec{x}_0}{\rho^2} - \frac{\left(\frac{a}{r_0}\right)^2\left[\left(\frac{r_0}{a}\right)^2\vec{x} - \vec{x}_0\right]}{\left(\frac{a}{r_0}\right)^2\rho^2}\right\}$$

$$= \frac{1}{2\pi\rho^2}\left[\vec{x}-\vec{x}_0 - \left(\frac{r_0}{a}\right)^2\vec{x} + \vec{x}_0\right].$$

Thus $\frac{\partial G}{\partial n} = \nabla G\cdot\hat{n} = \nabla G\cdot\frac{\vec{x}}{a} = \frac{1}{2\pi\rho^2}\left[\vec{x}-\vec{x}_0 - \left(\frac{r_0}{a}\right)^2\vec{x} + \vec{x}_0\right]\cdot\frac{\vec{x}}{a}$

$$= \frac{1}{2\pi\rho^2}\left[\frac{\vec{x}\cdot\vec{x}}{a} - \cancel{\frac{\vec{x}_0\cdot\vec{x}}{a}} - \left(\frac{r_0}{a}\right)^2\vec{x}\cdot\vec{x} + \cancel{\frac{\vec{x}_0\cdot\vec{x}}{a}}\right]$$

$$= \frac{1}{2\pi\rho^2}(a - r_0^2),\quad\text{since } \vec{x}\cdot\vec{x} = |\vec{x}|^2 = a^2.$$

$$\Rightarrow u(\vec{x}_0) = \int_{|\vec{x}|=a} h(\vec{x})\,G(\vec{x},\vec{x}_0)\,ds = \frac{a-|\vec{x}_0|^2}{2\pi}\int_{|\vec{x}|=a}\frac{h(\vec{x})}{\rho^2}\,ds$$

$$\boxed{u(\vec{x}_0) = \frac{a-|\vec{x}_0|^2}{\ }\int \frac{h(\vec{x})}{|\vec{x}-\vec{x}_0|^2}\,ds}$$

(17) a) Find the Green's function for the quadrant $Q = \{(x,y) : x>0, y>0\}$

b) Use your answer in part (a) to solve the Dirichlet problem
$$\begin{cases} u_{xx}+u_{yy}=0 & \text{in } Q \\ u(0,y) = g(y) & \text{for } y>0 \\ u(x,0) = h(x) & \text{for } x>0 \end{cases}$$

Soln:

Figure 1

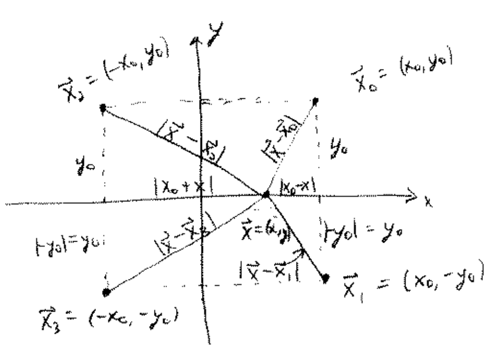
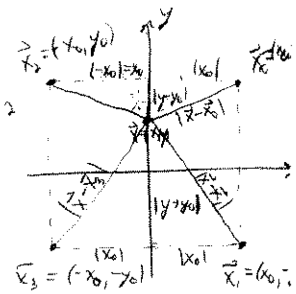


Figure 2



Claim: $G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \ln|\vec{x}-\vec{x}_0| - \frac{1}{2\pi} \ln|\vec{x}-\vec{x}_1| - \frac{1}{2\pi} \ln|\vec{x}-\vec{x}_2| + \frac{1}{2\pi} \ln|\vec{x}-\vec{x}_3|$.

We need to show that G satisfies (i)–(iii) as in (6).

(i) By the same computation as in (6) and (11), G has continuous second derivatives and $\Delta G = 0$ in D except at $\vec{x} = \vec{x}_0$ (this is a consequence of the fact that $\ln|\vec{x}-\vec{a}|$ is harmonic except at $\vec{x} = \vec{a}$).

(ii) By congruent triangles (see figures 1 and 2) we have the following 2 cases for $\vec{x} \in \partial Q$:

Case 1: $\vec{x} \in \partial Q$ such that $\vec{x} = (x,0)$ for $x>0$.

In this case (Figure 1), let $b = |\vec{x}-\vec{x}_0|$

Then $G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} (\ln b - \ln b - \ln b + \ln b) = 0$.

Case 2: $\vec{x} \in \partial Q$ such that $\vec{x} = (0,y)$ $y>0$.

Let $|\vec{x}-\vec{x}_0| = c$. Then (see Figure 2) we have

$G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} (\ln c - \ln c - \ln c + \ln c) = 0$.

(iii) $G(\vec{x}, \vec{x}_0) - \frac{1}{2\pi} \ln|\vec{x}-\vec{x}_0| = \frac{1}{2\pi} \left(-\ln|\vec{x}-\vec{x}_1| - \ln|\vec{x}-\vec{x}_2| + \ln|\vec{x}-\vec{x}_3| \right)$, which is finite at $\vec{x}_0$, has continuous second derivatives everywhere in Q, and is harmonic at $\vec{x}_0$ (this is a consequence of the fact that $\ln|\vec{x}-\vec{a}|$ is harmonic except at $\vec{x} = \vec{a}$ and the fact that $\vec{x}_1, \vec{x}_2$, and $\vec{x}_3$ are not equal to $\vec{x}_0$; in fact they are not in Q).

b) By Theorem 2 (p. 181),
$$u(\vec{x}_0) = \iint_{\partial Q} p(\vec{x}) \frac{\partial G(\vec{x}, \vec{x}_0)}{\partial n} \, da, \quad \text{where } p(\vec{x}) = \begin{cases} h(x) & \text{if } \vec{x} \in Q_1 \\ g(x) & \text{if } \vec{x} \in Q_2 \\ 0 & \text{otherwise} \end{cases}$$

where $Q_1 = \{(x,0) : x>0\}$ and $Q_2 = \{(0,y) : y>0\}$.

Then $\partial Q = Q_1 \cup Q_2 \cup \{0\}$. Now, if we denote the outward unit normals of Q_1 and Q_2 by $\hat{n}_1$ and $\hat{n}_2$, respectively, we have $\hat{n}_1 = -\hat{j}$ and $\hat{n}_2 = -\hat{i}$

Therefore $u(\vec{x}_0) = \int_{\partial Q} p(\vec{x}) \frac{\partial G(\vec{x}, \vec{x}_0)}{\partial n} \, da$

$$= \int_{Q_1} h(x) \left[-\frac{\partial G}{\partial y}(\vec{x}, \vec{x}_0) \right] da + \int_{Q_2} g(y) \left[-\frac{\partial G}{\partial x}(\vec{x}, \vec{x}_0) \right] da + \int_{\{0\}} 0 \, da$$

This integral vanishes regardless of the definition of p, since we are integrating from 0 to 0

$$= -\int_0^\infty h(x) \frac{\partial G}{\partial y}(x, 0, x_0, y_0) \, dx - \int_0^\infty g(y) \frac{\partial G}{\partial x}(0, y, x_0, y_0) \, dy$$

(17)

Now, $\frac{\partial G}{\partial x}(x,y,x_0,y_0) = \frac{\partial}{\partial x}\left[\frac{1}{2\pi}\left(\frac{1}{2}\ln|\vec{x}-\vec{x}_0|^2 - \frac{1}{2}\ln|\vec{x}-\vec{x}_1|^2 - \frac{1}{2}\ln|\vec{x}-\vec{x}_2|^2 + \frac{1}{2}\ln|\vec{x}-\vec{x}_3|^2\right)\right]$

$$= \frac{1}{4\pi}\frac{\partial}{\partial x}\left\{\ln\left[(x-x_0)^2+(y-y_0)^2\right] - \ln\left[(x-x_0)^2+(y+y_0)^2\right] - \ln\left[(x+x_0)^2+(y-y_0)^2\right] + \ln\left[(x+x_0)^2+(y+y_0)^2\right]\right\}$$

$$= \frac{1}{4\pi}\left[\frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2} - \frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2} + \frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2}\right]$$

$$\Rightarrow \frac{\partial G}{\partial x}(0,y,x_0,y_0) = \frac{1}{2\pi}\left[\frac{-x_0}{x_0^2+(y-y_0)^2} - \frac{-x_0}{x_0^2+(y+y_0)^2} - \frac{x_0}{x_0^2+(y-y_0)^2} + \frac{x_0}{x_0^2+(y+y_0)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2x_0}{x_0^2+(y-y_0)^2} + \frac{2x_0}{x_0^2+(y+y_0)^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{x_0}{x_0^2+(y+y_0)^2} - \frac{x_0}{x_0^2+(y-y_0)^2}\right]$$

Similarly

$$\frac{\partial G}{\partial y}(x,y,x_0,y_0) = \frac{1}{4\pi}\left[\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2} - \frac{2(y-y_0)}{(x+x_0)^2+(y-y_0)^2} + \frac{2(y+y_0)}{(x+x_0)^2+(y+y_0)^2}\right]$$

$$\Rightarrow \frac{\partial G}{\partial y}(x,0,x_0,y_0) = \frac{1}{2\pi}\left[\frac{-y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x-x_0)^2+y_0^2} - \frac{-y_0}{(x+x_0)^2+y_0^2} + \frac{y_0}{(x+x_0)^2+y_0^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-y_0}{(x-x_0)^2+y_0^2} + \frac{y_0}{(x+x_0)^2+y_0^2}\right]$$

$$\Rightarrow \boxed{u(\vec{x}_0) = \int_0^\infty h(x)\,\frac{1}{\pi}\left[\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x+x_0)^2+y_0^2}\right]dx + \int_0^\infty g(y)\,\frac{1}{\pi}\left[\frac{x_0}{x_0^2+(y-y_0)^2} - \frac{x_0}{x_0^2+(y+y_0)^2}\right]dy}$$

(18)

Class Exercises:

① We proved the Mean Value Property (MVP) for harmonic functions with concentric spherical averages in class. Prove the MVP for balls, i.e., if $B_R(x_0) \subset \Omega$, $\Delta u = 0$ in Ω, then

$$u(\vec{x}_0) = \int_{B_R(\vec{x}_0)} u(\vec{x})\,dV_n.$$

Hint: Let $v = R^2 - \|\vec{x}-\vec{x}_0\|^2$, use integration by parts.

Proof: Let u be a harmonic function in Ω, let $B_R(x_0) \subset \Omega$, and let $v = R^2 - \|\vec{x}-\vec{x}_0\|^2$. The integration by parts formula is

$$\int_{B_R^n(x_0)} (u\Delta v - v\underbrace{\Delta u}_{0})\,dV_n = \int_{S_R^{n-1}(x_0)} \left(u\frac{\partial v}{\partial n} - \underbrace{v}_{0 \text{ when } \|\vec{x}-\vec{x}_0\|=R}\frac{\partial u}{\partial n}\right)dV_{n-1}.$$

$$\Leftrightarrow \int_{B_R^n(x_0)} u\,\Delta v\,dV_n = \int_{S_R^{n-1}(\vec{x}_0)} u\frac{\partial v}{\partial n}\,dV_{n-1} \quad (*)$$

Now, $\left.\frac{\partial v}{\partial n}\right|_{S_R^{n-1}} = \nabla v \cdot \left.\frac{(\vec{x}-\vec{x}_0)}{R}\right|_{\|\vec{x}-\vec{x}_0\|=R} = -2(\vec{x}-\vec{x}_0)\cdot\left.\frac{(\vec{x}-\vec{x}_0)}{R}\right|_{\|\vec{x}-\vec{x}_0\|=R} = -2\left.\frac{\|\vec{x}-\vec{x}_0\|^2}{R}\right|_{\|\vec{x}-\vec{x}_0\|=R} = -2R$

since $\nabla(-\|\vec{x}-\vec{x}_0\|) = -\nabla\left[\sum_{i=1}^n (x_i-x_{0i})^2\right] = -2(x_1-x_0, x_2-x_0, \dots, x_n-x_0) = -2(\vec{x}-\vec{x}_0)$

Also, $\Delta v = \nabla\cdot(-2)(\vec{x}-\vec{x}_0) = -2\,\nabla\cdot(\vec{x}-\vec{x}_0) = -2\sum_{i=1}^n \frac{\partial}{\partial x_i}(x_i-x_{0i}) = -2\sum_{i=1}^n 1 = -2n$

Thus (*) becomes $-\int_{B_R^n(\vec{x}_0)} u\cdot 2n\,dV_n = -\int_{S_R^{n-1}(\vec{x}_0)} u\cdot 2R\,dV_{n-1}$

$$\Leftrightarrow n\int_{B_R^n(\vec{x}_0)} u(\vec{x})\,dV_n = R\int_{S_R^{n-1}(\vec{x}_0)} u(\vec{x})\,dV_{n-1}$$

We now divide both sides of the equation by $|B_1^n|\, R^n$, where $|W|$ denotes the volume of the set W.

This gives

$$\frac{n}{|B_1^n|\, R^n} \int_{B_R^n(\vec{x}_0)} u(\vec{x})\, dV_n = \frac{R}{|B_1^n|\, R^n} \int_{S_R^{n-1}(\vec{x}_0)} u(\vec{x})\, dV_{n-1}$$

$$\Leftrightarrow \frac{1}{|B_1^n|\, R^n} \int_{B_R^n(\vec{x}_0)} u(\vec{x})\, dV_n = \frac{1}{\underbrace{n|B_1^n|}_{\substack{|S_1^{n-1}| \\ \text{\#4}}}\, R^{n-1}} \int_{S_R^{n-1}(\vec{x}_0)} u(\vec{x})\, dV_{n-1}$$

$$= \frac{1}{|S_1^{n-1}|\, R^{n-1}} \int_{S_R^{n-1}(\vec{x}_0)} u(\vec{x})\, dV_{n-1}$$

$$= ⨍_{S_R^{n-1}(\vec{x}_0)} u(\vec{x})\, dV_{n-1}$$

$$= u(\vec{x}_0)$$

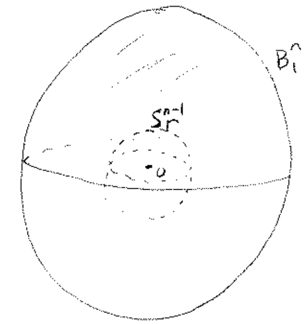
By the MVP for concentric spherical averages.

Since $\frac{1}{|B_1^n|\, R^n} \int_{B_R^n(\vec{x}_0)} u(\vec{x})\, dV_n = ⨍_{B_R^n(\vec{x}_0)} u(\vec{x})\, dV_n$, we have

$$⨍_{B_R^n(\vec{x}_0)} u(\vec{x})\, dV_n = u(\vec{x}_0).$$

\#4 We still need to show that $n|B_1^n| = |S_1^{n-1}|$.

Now, $|B_1^n| = \int_0^1 |S_1^{n-1}|\, r^{n-1}\, dr$. This is motivated by the following picture



Each sphere S_r^{n-1} inside B_1^n has surface area (volume) equal to $|S_1^{n-1}|\, r^{n-1}$, where r is the radius of S_r^{n-1}.

Then $|B_1^n| = \int_0^1 |S_r^{n-1}|\, dr = \int_0^1 |S_1^{n-1}|\, r^{n-1}\, dr$

$$= |S_1^{n-1}| \int_0^1 r^{n-1}\, dr = |S_1^{n-1}| \left.\frac{r^n}{n}\right|_0^1 = \frac{1}{n} |S_1^{n-1}|$$

$$\Rightarrow n|B_1^n| = |S_1^{n-1}|. \qquad \square$$

(5) Let $f \in C_c^2(\mathbb{R}^n)$
← compactly supported, i.e. $\overline{\{\vec{x} \mid f(\vec{x}) \neq 0\}}$ is bounded.

Define $u(\vec{x}) = \int_{\mathbb{R}^n} f(\vec{x}-\vec{\xi})\, K(\vec{\xi})\, d\vec{\xi} = f * K(\vec{x})$

a) Check $u(\vec{x}) = \int_{\mathbb{R}^n} f(\vec{\xi})\, K(\vec{x}-\vec{\xi})\, d\vec{\xi}$ as well $(f * K = K * f)$

b) Prove $\Delta u(\vec{x}) = f(\vec{x})$ (as part of this problem prove $u \in C^2(\mathbb{R}^n)$).

Proof: a) $u(\vec{x}) = \int_C f(\vec{x}-\vec{\xi})\, K(\vec{\xi})\, d\vec{\xi}$

where $C = \{\vec{x} \mid f(\vec{x}) \neq 0\}$.

Let $\tilde{\vec{x}} = \vec{x} - \vec{\xi}$
Then $\vec{\xi} = \vec{x} - \tilde{\vec{x}}$
and $d\vec{\xi} = |\det J|\, d\tilde{\vec{x}}$
$= |1|\, d\tilde{\vec{x}}$
$= d\tilde{\vec{x}}$

(i.e., translations don't change volume

So $u(\vec{x}) = \int_{\vec{x}-C} f(\vec{x}) K(\vec{x}-\vec{x})\, d\vec{x}$

where $\vec{x}-C = \{\vec{x} : f(\vec{x}) \neq 0\} = \{\vec{\xi} \in C : f(\vec{x}-\vec{\xi}) \neq 0\}$

So $u(\vec{x}) = \int_{\mathbb{R}^n} f(\vec{\xi}) K(\vec{x}-\vec{\xi})\, d\vec{\xi}$. $\square$

b) . (at end)

③ Show $n \geq 3$, $\Delta K(\vec{x}) = 0$ for $\vec{x} \neq 0$.

Hint: write $K(\vec{x}) = \alpha_n (\|\vec{x}\|^2)^{\frac{2-n}{2}}$

Proof: We have $\|\vec{x}\|^2 = \sum_{i=1}^{n} x_i^2$, where $\vec{x} \in \mathbb{R}^n$.

Now $\Delta K = \nabla\cdot(\nabla K)$, so

$$(\nabla K)_j = \alpha_n \cdot \frac{2-n}{2} (\|\vec{x}\|^2)^{\frac{2-n}{2}-1} \cdot 2x_j = \alpha_n (2-n)\, x_j\, (\|\vec{x}\|^2)^{-\frac{n}{2}}$$

and
$$\nabla\cdot(\nabla K) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (\nabla K)_j$$
$$= \sum_{j=1}^{n} \left\{ \alpha_n (2-n) \left[x_j^2 \cdot \left(-\frac{n}{2}\right) (\|\vec{x}\|^2)^{-\frac{n}{2}-1} \cdot 2x_j + (\|\vec{x}\|^2)^{-\frac{n}{2}} \right] \right\}$$
$$= \alpha_n (2-n) \sum_{j=1}^{n} \left[-n x_j^2 (\|\vec{x}\|^2)^{-\frac{n}{2}} (\|\vec{x}\|^2)^{-1} + (\|\vec{x}\|^2)^{-\frac{n}{2}} \right]$$
$$= \alpha_n (2-N) \left\{ \sum_{j=1}^{n} \left[-n x_j^2 \|\vec{x}\|^{-n} (\|\vec{x}\|^2)^{-1} \right] + n\|x\|^{-n} \right\}$$
$$= \alpha_n (2-n) \left[-n \|\vec{x}\|^{-n} (\|\vec{x}\|^2)^{-1} \sum_{j=1}^{n} x_j^2 + n\|\vec{x}\|^{-n} \right]$$
$$= \alpha_n (2-n) \left[-n \|\vec{x}\|^{-n} \cancel{(\|\vec{x}\|^2)^{-1} \|\vec{x}\|^2} + n\|\vec{x}\|^{-n} \right]$$
$$= 0. \quad \square$$

Class exercise 2b

if $f \in C_c^2(\mathbb{R}^n)$ show

$\Delta(f*K)(x) = f(x)$, where

($f \in C_c^2(\mathbb{R}^n)$ means f is C^2 & has compact support, so $\equiv 0$ outside a bounded set)

$$u(x) = f*K(x) = \int_{\mathbb{R}^n} f(x-y)K(y)\,dy$$

Step 1 $u \in C^2(\mathbb{R}^n)$ and

$$\Delta u = \int_{\mathbb{R}^n} \Delta_x(f(x-y))K(y)\,dy$$

proof We've shown $\int_C |K(y)|dy < \infty$ for any compact set.

This is because the singularity at the origin grows like $|y|^{2-n}$ $(n>2)$ or $|\ln y|$ $(n=2)$

So using n-dim'l polar coords $dV^n = (d\omega)^{n-1} r^{n-1} dr$
$\uparrow$
$\omega \in S_1$ the unit sphere

$$\int_{|y|\le R} |K(y)|\,dy = \int_0^R \underbrace{r^{2-n} r^{n-1}}_{r} |S_1|\,dr < \infty \qquad n \geq 3$$

$$= \int_0^R |\ln r| r \, 2\pi \, dr < \infty \qquad n=2$$

thus $\dfrac{u(\vec{x}+h\vec{e}_i)-u(\vec{x})}{h} = \displaystyle\int_{C} \underbrace{\frac{f(x+he_i-y)-f(x-y)}{h}}_{\substack{h\to 0 \\ \downarrow \\ \frac{\partial}{\partial x_i} f(x-y)}} K(y)\,dy$

uniformly on the bounded set on which $g(y) = f(x-y)$ is non-zero.

by e.g. the mean value property.

$$\frac{f(x+he_i-y)-f(x-y)}{h} = \frac{\partial f}{\partial x_i}(\xi - y)$$

$|\xi - x| < |h|$.

thus $\dfrac{\partial u}{\partial x_i} = \displaystyle\int_C \frac{\partial}{\partial x_i} f(x-y)K(y)\,dy$.

Similarly $\dfrac{\partial^2 u}{\partial x_i \partial x_j} = \displaystyle\int_C \frac{\partial^2}{\partial x_i \partial x_j} f(x-y)K(y)\,dy$.

Also similarly, $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$ is continuous.

Step 2 Show $\Delta_x u = f(x)$.

notice $\frac{\partial}{\partial x_i}(f(x-y)) = -\frac{\partial}{\partial y_i} f(x-y)$ chain rule

$$\frac{\partial^2 f}{\partial x_i^2} f(x-y) = +\frac{\partial^2}{\partial y_i^2} f(x-y)$$

So $$\Delta_x u = \int_{\mathbb{R}^n} \Delta_x (f(x-y)) K(y)\,dy = \int_{\mathbb{R}^n} \Delta_y (f(x-y)) K(y)\,dy$$

$$= \int_{B_\varepsilon(0)} \Delta_y (f(x-y)) K(y)\,dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} (\Delta_y f(x-y)) K(y)\,dy$$

$$= I_1 + I_2$$

I_1: Since $f \in C^2$ and since $\int_{B_\varepsilon(0)} |K(y)|\,dy \to 0$ as $\varepsilon \to 0$, deduce $I_1 \to 0$ as $\varepsilon \to 0$.

I_2: $$\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} (\Delta_y v) K - v \cancel{\Delta_y K}^{\,0} = -\Bigg[\underbrace{\int_{\partial B_\varepsilon} \frac{\partial v}{\partial n} K}_{} - \int_{\partial B_\varepsilon} v \frac{\partial K}{\partial n}\Bigg]$$

$$\downarrow \varepsilon \to 0$$

$$\int_{\mathbb{R}^n} \Delta_y (f(x-y)) K(y)\,dy$$

$| \; | \le C \int_{\partial B_\varepsilon} |K| \to 0$ as $\varepsilon \to 0$ since $|K| \le C \varepsilon^{2-n}$ (or $\varepsilon \ln \varepsilon$) and $|\partial B_\varepsilon| \le C \varepsilon^{n-1}$

$$\int_{\partial B_\varepsilon} \frac{\partial K}{\partial n} = 1$$ (that's how we normalized K)

so this second integral is the integral average of $f(x-y)$ over the y ε-sphere.

thus as $\varepsilon \to 0$ the right hand side converges to

$$- - f(x) = f(x)$$