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Math 5440
Homework #10

§5.27: 1, 3

1. $\begin{cases} u''-u=f(x) & x>0 \\ u(0)=u'(0)=0 \end{cases}$ $\qquad u(x)=\int_a^x R(x,\xi)f(\xi)\,d\xi \qquad a=0$

$R(x,\xi)=\frac{1}{pW}\left[v_2(x)v_1(\xi)-v_1(x)v_2(\xi)\right]$

$v''-v=0 \Rightarrow v=\{e^x, e^{-x}\} \quad v_1=e^x \quad v_2=e^{-x} \quad v_1'=e^x \quad v_2'=-e^{-x}$

$W=\begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix} = v_1v_2'-v_2v_1' = -e^xe^{-x}-e^xe^{-x}=-2 \quad p=1 \Rightarrow pW=-2$

$R=\frac{1}{2}\left[e^xe^{-\xi}-e^{-x}e^{\xi}\right]=\frac{1}{2}\left[e^{x-\xi}-e^{-(x-\xi)}\right]=\sinh(x-\xi)$

$\boxed{u(x)=\int_0^x \sinh(x-\xi)f(\xi)\,d\xi}$

3. $\begin{cases} u''+u'-2u=e^xF & x>0 \\ u(0)=1 \\ u'(0)=0 \end{cases}$ $\qquad a=1 \quad p=e^{\int^x \frac{b}{a}d\xi}=e^x$
$b=1 \quad q=\frac{c}{a}p=-2e^x$
$c=-2 \quad f=\frac{F}{a}p=e^{2x}$

$(e^xv')'-2e^xv=e^{2x}$

$v''+v'-2v=0 \qquad R=c_1e^{-2x}+c_2e^x$

$r^2+r-2=0 \quad r=-2,1$

$v=\operatorname{span}\{e^{-2x}, e^x\}$

$R(\xi,\xi)=0 \Rightarrow c_1e^{-2\xi}+c_2e^{\xi}=0 \Rightarrow c_2=-c_1e^{-3\xi}$

$R_x(\xi,\xi)=\frac{1}{p(\xi)} \Rightarrow -2c_1e^{-2\xi}+c_2e^{\xi}=e^{-\xi} \Rightarrow -2c_1e^{-2\xi}-c_1e^{-2\xi}=e^{-\xi}$
$=-3c_1e^{-2\xi}=e^{-\xi} \Rightarrow c_1=\frac{-e^{\xi}}{3} \quad c_2=\frac{e^{-2\xi}}{3}$

$R=\frac{-e^{\xi-2x}}{3}+\frac{e^{-2\xi+x}}{3}$

$u(x)=\int_0^x R(x,\xi)f(\xi)\,d\xi + c_1\cosh(x)$

$u_p(0)=1=0+c_1$

$u_h''+u_h'+2u_h=0$
$u_h(0)=1$
$u_h'(0)=0$

$u_p(x)=\int_0^x R(x,\xi)f(\xi)\,d\xi$
$=\frac{1}{3}\int_0^x -e^{(\xi-2x)}\cdot e^{2\xi}+e^{-2\xi+x}\cdot e^{2\xi}\,d\xi$
$=\frac{1}{3}\int_0^x -e^{3\xi-2x}+e^x\,d\xi$
$=\frac{1}{3}\left[\frac{-1}{3}e^{3\xi-2x}+e^x\xi\right]_0^x$
$=\frac{1}{3}\left[-\frac{1}{3}(e^x-e^{-2x})+xe^x\right]$
$=\frac{1}{9}e^{-2x}-\frac{1}{9}e^x+\frac{1}{3}xe^x$

$u_h=c_1e^{-2x}+c_2e^x$
$u_h'=-2c_1e^{-2x}+c_2e^x$
$c_1+c_2=1 \Rightarrow c_2=1-c_1$
$-2c_1+c_2=0 \Rightarrow -2c_1+1-c_1=0$
$3c_1=1 \quad c_1=1/3$
$c_2=2/3$

$u(x)=\frac{1}{9}e^{-2x}-\frac{1}{9}e^x+\frac{1}{3}xe^x-\frac{1}{3}e^{-2x}+\frac{2}{3}e^x$
$=\frac{4}{9}e^{-2x}+\frac{5}{9}e^x+\frac{1}{3}xe^x$

$\boxed{u(x)=\frac{1}{3}\left(x+\frac{5}{3}\right)e^x+\frac{4}{9}e^{-2x}}$

§5.28: 1,2

1. $\begin{cases} u''-u = -f(x) & 0<x<1 \\ u(0)=u(1)=0 \end{cases}$ $\qquad u(x)=\int_\alpha^\beta G(x,\xi)f(\xi)\,d\xi$

$G''-G=0 \Rightarrow G=\text{span}\{\sinh x, \cosh x\}$

$G_1 = c_1\sinh(x) \quad x<\xi \qquad p=1 \Rightarrow \partial_x G_1|_{x=\xi} = \partial_x G_2|_{x=\xi} + 1$

$G_2 = c_2\sinh(x-1) \quad x>\xi \qquad c_1\cosh(\xi) = c_2\cosh(\xi-1) + 1$

$G_1(\xi,\xi)=G_2(\xi,\xi) \qquad c_1\sinh(\xi) = c_2\sinh(\xi-1)$

$$\begin{bmatrix}\cosh\xi & -\cosh(\xi-1)\\ \sinh\xi & -\sinh(\xi-1)\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \frac{1}{-\cosh\xi\sinh(\xi-1)+\sinh\xi\cosh(\xi-1)}\begin{bmatrix}-\sinh(\xi-1)\\ -\sinh(\xi)\end{bmatrix} = \frac{1}{\sinh(1)}\begin{bmatrix}-\sinh(\xi-1)\\ -\sinh(\xi)\end{bmatrix}$$

$$u(x)=\frac{1}{\sinh(1)}\left[\sinh(x)\int_x^1 \sinh(1-\xi)f(\xi)\,d\xi + \sinh(1-x)\int_0^x \sinh(\xi)f(\xi)\,d\xi\right]$$

2. $\begin{cases} u''-u = -f(x) & 0<x<1 \\ u'(0)=u'(1)=0 \end{cases}$ $\qquad u(x)=\int_0^1 G(x,\xi)f(\xi)\,d\xi$

$G''-G=0 \quad G=\text{span}\{\sinh x, \cosh x\}$

$G_1 = c_1\cosh x \quad x<\xi \qquad p=1 \Rightarrow \partial_x G_1|_{x=\xi} = \partial_x G_2|_{x=\xi} + 1$

$G_2 = c_2\cosh(x-1) \quad x>\xi \qquad c_1\sinh(\xi) = c_2\sinh(\xi-1) + 1$

$G_1(\xi,\xi)=G_2(\xi,\xi) \qquad c_1\cosh(\xi) = c_2\cosh(\xi-1)$

$$\begin{bmatrix}\sinh(\xi) & -\sinh(\xi-1)\\ \cosh(\xi) & -\cosh(\xi-1)\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \frac{1}{-\sinh\xi\cos(\xi-1)+\cosh(\xi)\sinh(\xi-1)}\begin{bmatrix}-\cosh(\xi-1)\\ -\cosh(\xi)\end{bmatrix} = \frac{1}{\sinh(1)}\begin{bmatrix}\cosh(\xi-1)\\ \cosh(\xi)\end{bmatrix}$$

$$u(x)=\frac{1}{\sinh(1)}\left[\cosh x\int_x^1 \cosh(1-\xi)f(\xi)\,d\xi + \cosh(1-x)\int_0^x \cosh(\xi)f(\xi)\,d\xi\right]$$

Laurel Terms

§5.29: 1, 2, 4, 7

1. $\Delta u = -y(y-1)\sin^3 x \quad 0<x<\pi \quad 0<y<1$

$u(x,0) = u(x,1) = u(0,y) = u(\pi,y) = 0$

$F(x,y) = y(y-1)\left[\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right] = \frac{A_0}{2} + \sum A_n \cos nx + \sum B_n \sin nx$

$B_n = 0$ for $n \neq 1,3$ $\quad B_1 = \frac{3}{4}y(y-1) \quad B_3 = -\frac{1}{4}y(y-1) \quad A_n = 0,\ A_0 = 0$

$u(x,y) \sim \frac{a_0(y)}{2} + \sum a_n(y)\cos nx + \sum b_n(y)\sin nx$

$(\partial_x^2 + \partial_y^2)u \sim \sum_{n=1}^{\infty}(-n^2 b_n(y) + b_n''(y))\sin nx = \sum -B_n(y)\sin nx$

$\begin{cases} b_1'' - b_1 = -\frac{3}{4}y(y-1) \\ b_1(0) = b_1(1) = 0 \end{cases} \qquad \begin{cases} b_3'' - 9b_3 = \frac{1}{4}y(y-1) \\ b_3(0) = b_3(1) = 0 \end{cases}$

$n=1$: $G'' - G = 0 \quad G = \text{span}\{\sinh y, \cosh y\}$

$G_1 = c_1 \sinh y \quad x<\xi \qquad p=1 \Rightarrow \partial_y G_1|_{y=\xi} = \partial G_2|_{y=\xi} + 1$

$G_2 = c_2 \sinh(y-1) \quad x>\xi \qquad c_1\cosh\xi - c_2\cosh(\xi-1) = 1$

$G_1(\xi,\xi) = G_2(\xi,\xi) \qquad c_1\sinh\xi - c_2\sinh(\xi-1) = 0$

$\begin{bmatrix} \cosh\xi & -\cosh(\xi-1) \\ \sinh\xi & -\sinh(\xi-1) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sinh 1}\begin{bmatrix} \sinh(1-\xi) \\ -\sinh(\xi) \end{bmatrix}$

$b_1 = \frac{-3/4}{\sinh(1)}\left[\sinh y \int_y^1 \sinh(\xi-1)\cdot(\xi^2-\xi)\,d\xi + \sinh(y-1)\int_0^y \sinh(\xi)\cdot(\xi^2-\xi)\,d\xi\right]$

$= \frac{-3/4}{\sinh(1)}\left[\sinh y\left[(\xi^2-\xi)\cosh(\xi-1) - (2\xi-1)\sinh(\xi-1) + 2\cosh(\xi-1)\right]_y^1\right.$

$\left. + \sinh(y-1)\left[(\xi^2-\xi)\cosh\xi - (2\xi-1)\sinh\xi + 2\cosh\xi\right]_0^y\right]$

$= \frac{-3}{4\sinh(1)}\left[\sinh y\left[-(y^2-y)\cosh(y-1) - \cancel{\sinh(0)} + \cancel{(2y-1)\sinh(y-1)} + 2 - 2\cosh(y-1)\right]\right.$

$\left. + \sinh(y-1)\left[(y^2-y)\cosh y - \cancel{(2y-1)\sinh y} + 2\cosh y - 2\right]\right]$

$= \frac{-3}{4\sinh 1}\left[(y^2-y)\sinh(-1) - 2\sinh(1) + 2\sinh y - 2\sinh(y-1)\right]$

$\boxed{b_1 = \frac{3}{4}\left[-(y-y^2) + 2\left(1 - \frac{\cosh(y-1/2)}{\cosh(1/2)}\right)\right]}$

$n=3 \quad G'' - 9G = 0 \qquad G = \text{span}\{\sinh 3y, \cosh 3y\}$

$\begin{bmatrix} 3\cosh 3\xi & -3\cosh 3(\xi-1) \\ \sinh 3\xi & -\sinh 3(\xi-1) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad 3c_1\cosh 3\xi - 3c_2\cosh 3(\xi-1) = 1$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{-1}{3\sinh(3)}\begin{bmatrix} +\sinh 3(\xi-1) \\ +\sinh 3\xi \end{bmatrix} \qquad c_1\sinh 3\xi - c_2\sinh 3(\xi-1) = 0$

$b_3 = \frac{+1}{12\sinh 3}\left[\sinh 3y \int_y^1 \sinh 3(\xi-1)(\xi^2-\xi)\,d\xi + \sinh 3(y-1)\int_0^y \sinh 3\xi\,(\xi^2-\xi)\,d\xi\right]$

$= \frac{+1}{12\sinh 3}\left[\sinh 3y\left[\frac{1}{3}(\xi^2-\xi)\cosh 3(\xi-1) - \frac{1}{9}(2\xi-1)\sinh 3(\xi-1) + \frac{2}{27}\cosh 3(\xi-1)\right]_y^1\right.$

$\left. + \sinh 3(y-1)\left[\frac{1}{3}(\xi^2-\xi)\cosh 3(\xi) - \frac{1}{9}(2\xi-1)\sinh 3(\xi) + \frac{2}{27}\cosh 3\xi\right]_0^y\right]$

$= \frac{+1}{12\sinh 3}\left[\sinh 3y\left[-\frac{1}{3}(y^2-y)\cosh 3(y-1) + \cancel{\frac{1}{9}(2y-1)\sinh 3(y-1)} + \frac{2}{27} - \frac{2}{27}\cosh 3(y-1)\right)\right.$

$\sinh 3(y-1)\left(\frac{1}{3}(y^2-y)\cosh 3y - \cancel{\frac{1}{9}(2y-1)\sinh 3y} + \frac{2}{27}\cosh 3y - \frac{2}{27}\right)$

$= \frac{1}{12\sinh 3}\left[\frac{1}{3}(y^2-y)\sinh(-3) + \frac{2}{27}\sinh(-3) + \frac{2}{27}\sinh 3y - \frac{2}{27}\sinh 3(y-1)\right]$

$= -\frac{1}{4}\left[\frac{-1}{9}(y-y^2)\sinh(3) + \frac{2}{81}\left(1 - \frac{\cosh 3(y-1/2)}{\cosh 3/2}\right)\right]$

b_3

$\boxed{u(x,y) = b_1\sin x + b_3\sin 3x}$

2. $\begin{cases} \nabla^2 u = -2 & x^2+y^2<R^2 \\ u=0 & x^2+y^2=1 \end{cases}$

$= \begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -2 & \text{for } r<R \\ u(0) & \text{for } r=R \end{cases}$

$= (r u_r)_r + \frac{u_{\theta\theta}}{r} = -2r = F(r,\theta)\cdot r$

$F(r,\theta) = \frac{A_0}{2} + \sum A_n \cos n\theta + \sum B_n \sin n\theta \Rightarrow A_n = B_n = 0 \quad A_0 = 4$

$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(r)\cos n\theta + \sum_{n=1}^{\infty} b_n(r)\sin n\theta$

$(r u_r)_r = \frac{(r a_0')_r}{2} + \sum_{n=1}^{\infty}(r a_n')_r \cos n\theta + \sum_{n=1}^{\infty}(r b_n')_r \sin n\theta$

$\frac{1}{r}u_{\theta\theta} = \sum -\frac{n^2}{r} a_n \cos n\theta + \sum -\frac{n^2}{r} b_n \sin n\theta$

$L(u) = \frac{(r a_0')_r}{2} + \sum \left[(r a_n')_r - \frac{n^2 a_n}{r}\right]\cos n\theta + \sum\left[(r a_n')_r - \frac{n^2}{r}a_n\right]\sin n\theta$

$= -\frac{r A_0}{2} = -2r$

$(r a_0')_r = -4r \qquad a_0(R) = 0$

$\left[(r a_n')_r - \frac{n^2 a_n}{r}\right] = 0 \qquad a_n(R) = 0 \quad a_n(0) \text{ bdd}$

$\left[(r b_n')_r - \frac{n^2 b_n}{r}\right] = 0 \qquad b_n(R) = 0 \quad b_n(0) \text{ bdd}$

$\Rightarrow a_n = b_n = 0$

$(r a_0') = \int_0^r -4r\,dr = \frac{-4}{2}r^2$

$a_0' = \frac{1}{r}2r^2$

$a_0 = \int_r^a 2r\,dr = r^2\Big]_r^a = a^2 - r^2 \qquad a_0(R) = 0 \Rightarrow a = R$

$\boxed{u(r,\theta) = \frac{1}{2}\left[R^2 - r^2\right]}$

4. $\nabla^2 u = -F(x,y) \qquad 0<x<\pi \qquad 0<y<A$

$u(x,0) = u(x,A) = u(0,y) = u(\pi,y) = 0$

$F(x,y) \sim \sum_{n=1}^{\infty} B_n(y) \sin nx \qquad B_n(y) = \frac{2}{\pi}\int_0^{\pi} F(x,y)\sin nx\, dx$

$u(x,y) \sim \sum_{n=1}^{\infty} b_n(y) \sin nx$

$b_n(0) = b_n(A) = 0$

$(\partial_x^2 + \partial_y^2)u \sim \sum_{n=1}^{\infty} (-n^2 b_n + b_n''(y)) \sin nx = \sum -B_n(y) \sin nx$

$$\Rightarrow \begin{cases} b_n'' - n^2 b_n = -B_n(y) \\ b_n(0) = b_n(A) = 0 \end{cases}$$

$\Rightarrow b_n = \int_0^A G(y,\xi) B_n(\xi)\, d\xi$

$G_n'' - n^2 G_n = 0 \quad \Rightarrow \quad g_n = \mathrm{span}\{\sinh(ny), \cosh(ny)\}$

$G_{n1} = C_1 \sinh(ny) \quad y<\xi \qquad G_{n2} = C_2 \sinh n(y-A) \quad n>\xi$

$\partial_y G_{n_1}|_{y=\xi} = \partial_y G_{n_2}|_{y=\xi} + 1 \Rightarrow \quad nc_1 \cosh(n\xi) - nc_2 \cosh(\xi - A) = 1$

$G_1(\xi,\xi) = G_2(\xi,\xi) \quad \Rightarrow \quad c_1 \sinh(n\xi) - c_2 \sinh(\xi - A) = 0$

$\begin{bmatrix} n\cosh(n\xi) & n\cosh n(\xi-A) \\ \sinh n\xi & -\sinh(\xi-A) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \frac{1}{\begin{matrix}-n\cosh(n\xi)\sinh(\xi-A) \\ +n\cosh(\xi-A)\sinh(\xi)\end{matrix}} \begin{bmatrix} -\sinh n(\xi-A) \\ -\sinh(n\xi) \end{bmatrix} = \frac{1}{n\sinh(nA)} \begin{bmatrix} -\sinh n(\xi-A) \\ -\sinh(n\xi) \end{bmatrix}$

$b_n = \frac{-\sinh n(A-y)}{n\sinh(nA)} \int_0^y + \sinh(n\xi) B_n(\xi)\, d\xi - \frac{\sinh ny}{n\sinh(nA)} \int_y^1 \sinh n(\xi-A)\, B_n(\xi)\, d\xi$

$b_n = \frac{2\sinh n(y-A)}{\pi n \sinh(nA)} \int_0^y \sinh(n\xi) \int_0^{\pi} F(x,\xi) \sin nx\, dx\, d\xi + \frac{2\sinh ny}{n\sinh(nA)} \int_y^1 \sinh n(A-\xi) \int_0^{\pi} F(x,\xi) \sin nx\, dx\, d\xi$

$$\boxed{\begin{aligned} b_n &= \frac{2}{\pi n \sinh(nA)} \int_0^{\pi} \left\{ \sinh n(y-A) \int_0^y \sinh(n\xi) F(x,\xi)\, d\xi + \sinh ny \int_y^1 \sinh n(A-\xi) F(x,\xi)\, d\xi \right\} \sin nx\, dx \\ u(x,y) &= \sum_{n=1}^{\infty} b_n \sin nx \end{aligned}}$$

7. $\begin{cases} u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = -F(r,\theta) & r<R \\ u_r(R,\theta) = 0 \end{cases}$

$r\,u_{rr} + u_r + \frac{u_{\theta\theta}}{r} = -rF(r,\theta) = (ru_r)_r + \frac{1}{r}u_{\theta\theta}$

$F(r,\theta) \sim \frac{A_0(r)}{2} + \sum_{n=1}^{\infty} A_n(r)\cos n\theta + \sum_{n=1}^{\infty} B_n(r)\sin n\theta$

$u(r,\theta) = \frac{a_0}{2}(r) + \sum_{n=1}^{\infty} a_n(r)\cos n\theta + \sum_{n=1}^{\infty} b_n(r)\sin n\theta$

$ru_r = \frac{ra_0'}{2} + \sum_{n=1}^{\infty} ra_n'\cos n\theta + \sum_{n=1}^{\infty} rb_n'\sin n\theta$

$(ru_r)_r = \frac{(ra_0')'}{2} + \sum_{n=1}^{\infty}(ra_n')_r\cos n\theta + \sum_{n=1}^{\infty}(rb_n')_r\sin n\theta$

$\frac{1}{r}u_{\theta\theta} = \sum -\frac{n^2}{r}a_n\cos n\theta + \sum -\frac{n^2}{r}b_n\sin n\theta$

$L(u) = \frac{(ra_0')_r}{2} + \sum\left[(ra_n')_r - \frac{n^2a_n}{r}\right]\cos n\theta + \sum\left[(rb_n')_r - \frac{n^2b_n}{r}\right]\sin n\theta$

$= -\frac{rA_0}{2} - \sum rA_n\cos n\theta - \sum rB_n\sin n\theta$

$(ra_0')_r = -rA_0 \qquad a_0'(R) = 0$

$(ra_n')_r - \frac{n^2a_n}{r} = -rA_n \qquad a_n'(R) = 0 \qquad a_n(r)$ bdd near $n=0$

$(rb_n')_r - \frac{n^2b_n}{r} = -rB_n \qquad b_n'(R) = 0 \qquad b_n(r)$ bdd near $r=0$

$n \geq 1 \qquad G_n' + rG_n'' - \frac{n^2G_n}{r} = 0 \Rightarrow G = \text{span}\{r^{-n}, r^n\}$

$G_1 = c_1 r^n$

$G_2 = c_3 r^{n-1} - c_4 r^{-n-1}$

$c_3 R^{n-1} - c_4 R^{-n-1} = 0$

$\frac{c_3}{c_4} = R^{-2n} \Rightarrow G_2 = C_2\left(\left(\frac{R}{r}\right)^n + \left(\frac{r}{R}\right)^n\right)$

$\partial_r G_1|_{r=\xi} = \partial_r G_2|_{r=\xi} + \frac{1}{r(\xi)} \Rightarrow C_1 n\,\xi^{n-1} - C_2\left(\frac{-nR^n}{\xi^{n+1}} + \frac{n\xi^{n-1}}{R^n}\right) = \frac{1}{\xi}$

$G_1(\xi,\xi) = G_2(\xi,\xi) \qquad C_1\xi^n - C_2\left(\left(\frac{R}{\xi}\right)^n + \left(\frac{\xi}{R}\right)^n\right) = 0$

$\begin{bmatrix} n\xi^{n-1} & \frac{n}{\xi}\left[\left(\frac{R}{\xi}\right)^n - \left(\frac{\xi}{R}\right)^n\right] \\ \xi^n & -\left(\frac{R}{\xi}\right)^n - \left(\frac{\xi}{R}\right)^n \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\xi} \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{2nR^n}\begin{bmatrix} \left[\left(\frac{R}{\xi}\right)^n + \left(\frac{\xi}{R}\right)^n\right] \\ \xi^n \end{bmatrix}$

$$\boxed{\begin{aligned} a_n(r) &= \frac{1}{2nR^2}\left\{\left[\left(\frac{R}{r}\right)^n + \left(\frac{r}{R}\right)^n\right]\int_0^r \xi^{n+1}A_n(\xi)\,d\xi + r^n\int_r^R\left[\left(\frac{R}{\xi}\right)^n + \left(\frac{\xi}{R}\right)^n\right]\xi A_n(\xi)\,d\xi\right\} \\ b_n(r) &= \frac{1}{2nR^2}\left\{\left[\left(\frac{R}{r}\right)^n + \left(\frac{r}{R}\right)^n\right]\int_0^r \xi^{n+1}B_n(\xi)\,d\xi + r^n\int_r^R\left[\left(\frac{R}{\xi}\right)^n + \left(\frac{\xi}{R}\right)^n\right]\xi B_n(\xi)\,d\xi\right\} \end{aligned}}$$

$(ra_0')_r = -rA_0$

$Ra_0'(R) - ra_0'(r) = \int_r^R -\bar{r}A_0(\bar{r})\,d\bar{r} \qquad dv = \frac{-1}{r} \quad u = \int_0^r A_0\bar{r} \quad du =$

$-a_0'(r) = \frac{1}{r}\int_r^R \bar{r}A_0\,d\bar{r} + \cancel{\frac{1}{r}Ra_0'(R)}^{\,0} = \frac{-1}{r}\int_0^r \bar{r}A_0\,d\bar{r}$

$$\boxed{\iint_{\partial u}\nabla u\cdot n = 0 = \iint \Delta u = \iint_D F\,dA = \int_0^R\int_0^{2\pi} r\cdot F\,dr\,d\theta = 0 = 2\pi\int_0^R \frac{A_0}{2}r\,dr}$$

$$\boxed{a_0(r)' = -\ln(r)\int^r \bar{r}A_0\,d\bar{r} + \int^r A_0\bar{r}\ln(\bar{r})\,d\bar{r} = \int_0^r A_0\bar{r}\ln\left(\frac{r}{\bar{r}}\right)d\bar{r} = a_0(r)}$$

$$\boxed{u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\theta + \sum_{n=1}^{\infty} b_n\sin n\theta}$$