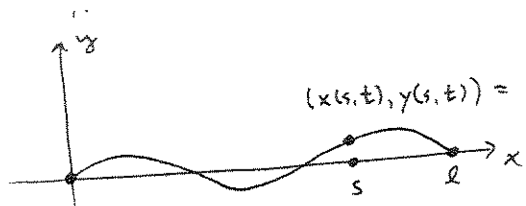
Math 5440
Monday Aug 23

- syllabus
- Evans PDE sample sheet

§1.1 : Derivations of the one space dimension wave equation

<u>vibrating string</u>:



$\vec{r}(0,t) = \begin{bmatrix} x(0,t) \\ y(0,t) \end{bmatrix} \equiv \vec{0} \quad t \geq 0$ fixed endpoints

$\vec{r}(\ell,t) = \begin{bmatrix} x(\ell,t) \\ y(\ell,t) \end{bmatrix} \equiv \begin{bmatrix} \ell \\ 0 \end{bmatrix} \quad t \geq 0$

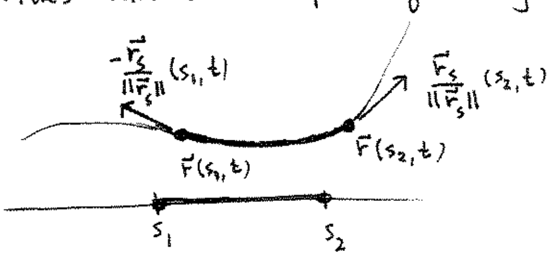
$\vec{r}(s,0) = \vec{r}_0(s)$ initial position specified.
$\vec{r}_t(s,0) = \vec{v}_0(s)$ initial velocity

possible IBVP

Physics:

$\rho(s) \quad 0 \leq s \leq \ell$ initial density mass/length
$T(s,t)$ tension force in string at $\vec{r}(s,t)$

Newtons law on test piece of string : D_t (momentum) = net forces



Force/mass along string

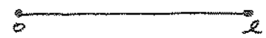
$$D_t \int_{s_1}^{s_2} \rho(s)\, \vec{r}_t(s,t)\, ds = T \frac{\vec{r}_s}{\|\vec{r}_s\|} \Big]_{(s_1,t)}^{(s_2,t)} + \int_{s_1}^{s_2} \rho(s)\, \vec{F}(s,t)\, ds$$

$$\overset{\|}{\int_{s_1}^{s_2}} \rho(s)\, \vec{r}_{tt}(s,t)\, ds$$

complicated system of PDE's!

Fix s_1, let s_2 vary, take $\frac{d}{ds_2}$, then subs $s = s_2$:
$$\boxed{\rho\, \vec{r}_{tt} = \left(T \frac{\vec{r}_s}{\|\vec{r}_s\|}\right)_s + \rho \vec{F}}$$

Assume

$\vec{r}(s,t) = (s, 0)$ is an equilibrium sol'n 

in this case tension T must be constant $T \equiv T_0$, else string would move.

$\boxed{\text{Case 1}}$ $x(s,t) = s$ technically there needs to be a small error term, see text or later notes

$y(s,t) = y(x,t)$ small $(|y|, |y_x|, |y_t| < \varepsilon)$

$\boxed{\text{transverse oscillations}}$

vert. comp of page 1 system:

$$\rho y_{tt} = \left(T \frac{y_x}{\sqrt{1+y_x^2}}\right)_x + \rho f_2 \qquad \text{if } \vec{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

assume $T \equiv T_0$ (approximately true if T depends on stretch per unit length since $\|\vec{r}_x\| = \sqrt{1+y_x^2} = 1 + O(\varepsilon^2)$.

so to within $O(\varepsilon^2)$,

$$\rho y_{tt} = T_0 y_{xx} + \rho f_2$$

so $y(x,t)$ satisfies $\boxed{\text{1-d wave equation}}$

$$\text{IBVP} \begin{cases} \boxed{u_{tt} = c^2 u_{xx} + f} & \boxed{c = \sqrt{\frac{T_0}{\rho}},\ f = f_2} \\ u(x,0) = u_0(x) & \text{init disp} \\ u_t(x,0) = v_0(x) & \text{init vel.} \\ u(x,0) = u(\ell,t) = 0 \quad t \geq 0 & \text{fixed ends} \end{cases}$$

<u>Remarks</u> As with linear DE's the general sol'n to the linear PDE wave eqtn is a particular solt'n plus the general sol'n to the homog. wave eqtn

$$\boxed{u_{tt} = c^2 u_{xx}} \qquad \text{homog. wave eqtn}$$

(you can find a time independent sol'tn to the inhomogeneous eqtn, i.e. a fcn of x alone, by antidifferentiating

$$0 = c^2 u_{xx} + f$$

twice wrt x.) (assuming f only depends on x.)

If c is constant, then for any fcn $U(z)$ of one variable,

$$u(x,t) = U(x-ct)$$
$$v(x,t) = U(x+ct)$$

solve $\boxed{u_{tt} = c^2 u_{xx}}$

check! :

For this reason, c is called the <u>speed</u>, since for $t = t_1$ the graph $y = U(x-ct_1)$ is the graph of $y = U(x)$ moved ct_1 units to the right ($y = U(x+ct_1)$ moves ct_1 to left)

Slinky experiment!

m = mass = 225 g

slinky has 89 loops

≈ 2.5 g/loop.

- find ℓ so that tension $T_0 = .2g$
- thus $\rho_0 = \frac{m}{\ell}$

$$c = \sqrt{\frac{T_0}{\rho_0}} = \sqrt{\frac{T_0 \ell}{m}}$$

- compare to actual wave speed!

Case II Longitudinal vibration, small oscillations

set $y(s,t) \equiv 0$ in page 1 system.

then 1st component yields

$$\boxed{\rho x_{tt} = \left(T \frac{x_s}{|x_s|}\right)_s + \rho f_1} \qquad \vec{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

since $x(s,t) \approx s$,

$$\frac{x_s}{|x_s|} \equiv 1$$

Assume equilibrium $\rho(s) \equiv \rho_0$ constant.

Assume tension depends on density $\rho(s,t) = \frac{\rho_0}{x_s}$ ($\rho_0 ds = \rho(s,t)\, x_s ds$ by conservation of mass)

$$T(s,t) = \tau\left(\frac{\rho_0}{x_s}\right)$$

Assume $|x(s,t) - s|$, $|x_s - 1|$, $|x_t|$, $|x_{ss}|$, $|x_{tt}|$ are $\le \varepsilon$ (small)

$$\rho_0 x_{tt} = \underbrace{\left(\tau\left(\frac{\rho_0}{x_s}\right)\right)_s}_{} + \rho_0 f_1$$

$$= \underbrace{\tau'\left(\frac{\rho_0}{x_s}\right)\left(-\frac{\rho_0 x_{ss}}{x_s^2}\right)}_{}$$

$$= \left(\tau'(\rho_0) + O(\varepsilon)\right)\left(-\rho_0 x_{ss}\right)\left(1 + O(\varepsilon)\right) + \rho_0 f_1$$

"Order ε"

$O(\varepsilon)$ denotes an error term bounded uniformly by $C\varepsilon$ for some fixed C

eliminate $O(\varepsilon^2)$ terms:

$$\rho_0 x_{tt} = -\rho_0 \tau'(\rho_0)\, x_{ss} + \rho_0 f_1$$

$$\boxed{x_{tt} = c^2 x_{ss} + f}$$

Wave eqtn!

maybe different speed $c = \sqrt{-\tau'(\rho_0)}$

than tranverse wave speed $c = \sqrt{\frac{T_0}{\rho_0}}$

Remark If a slinky is modeled as a Hooke's spring with rest length 0,

then $T = k\ell$

$\uparrow$

Hooke's const

$\rho = \frac{m}{\ell}$ so $\tau(\rho) = \frac{km}{\rho}$

so $\tau'(\rho) = -\frac{km}{\rho^2}$, $-\tau'(\rho_0) = \frac{T_0}{\rho}$ same speed as for transverse

Test!

Homework for Friday August 27

0) Read text's more rigorous explanation for transverse oscillations of a vibrating string leading to wave eqtn. Also note their somewhat different model for longitudinal vibrations, (p. 6a), 1-d motion in elastic solid.

1) Derive equations from §1.1 of text:

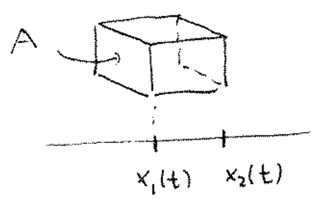
$$(1.13)\quad \begin{cases} \rho u_x + u\rho_x + \rho_t = 0 \\ \rho u_t + \rho u u_x + p_x = \rho F \end{cases}$$

for small disturbance of a gas in the x-direction (plane waves in $\mathbb{R}^3$)

$\rho(x,t) =$ mass density at (x,y,z,t)
$u(x,t) =$ x-velocity at (x,y,z,t) (y,z vel $= 0$).

as text says, 1st eqtn is conservation of mass
2nd is Newton's 2nd law (D_t(momentum) = net forces)

you may wish to consider a test box moving with the velocity flow:



recall that pressure is the magnitude of force/area that is exerted in the normal direction of a surface.

$x_1'(t) = u(x_1(t), t)$
$x_2'(t) = u(x_2(t), t)$

2) Assuming $\rho = \rho_0 + O(\varepsilon)$
$\rho_x, \rho_t = O(\varepsilon)$
$u, u_t, u_x = O(\varepsilon)$
$F = O(\varepsilon)$

derive linearization

$$1.14\quad \begin{cases} \rho_0 u_x + \rho_t = 0 \\ \rho_0 u_t + p'(\rho_0)\rho_x = \rho_0 F \end{cases}$$

3) Expand text details to yield

$$u_{tt} - c^2 u_{xx} = F_t \qquad c^2 = p'(\rho_0)$$
$$\rho_{tt} - c^2 \rho_{xx} = -\rho_0 F_x$$

(notice analogy to slinky eqtn from class)

4) Sound waves in air. Use internet (wiki) to verify that a fixed (enclosed) volume V of air with mass m at pressure p satisfies

$$* \quad pV^{\gamma} = C \qquad (C \text{ proportional to absolute temperature}).$$

$\gamma \approx 1.4$ for air is the adiabatic constant

Since $\rho = \frac{m}{V}$ you can compute $p'(\rho)$ for c = (3), using *
At 1 atmosphere of pressure and temperature 25° Celsius (one possible STP)
use these relations to model the speed of sound, i.e. get a theoretical value.
Compare with the value given in a table of actual values.
Include references with your answers.