Math 4200-001
Week 3 concepts and homework
1.5 - 1.6
Due Wednesday September 16 at 5:00 p.m.

1.5 25, 26, 27, 28, 31.
1.6 1c, 2abc, 3a, 4, 5.

extra credit (5 points) As we discuss in class on Friday Sept 11, a real-differentiable map $F : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is conformal, i.e. preserves angles between tangent vectors, and which also preserves orientation must be a rotation dilation. Prove this.

Hints: For each tangent vector $\gamma'(t_0) = \vec{v} \in T_{(x_0, y_0)}\mathbb{R}^2$, and writing $F(x, y) = (u(x, y), v(x, y))$, the differential map is given by

$$dF_{(x_0, y_0)}(\vec{v}) = (F \circ \gamma)'(t_0)$$

and the multivariable chain rule says we can compute this by the formula which uses the differential (aka derivative or Jacobian) matrix:

$$dF_{(x_0, y_0)}(\vec{v}) = DF(x_0, y_0)\,\vec{v} = \begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Your job is to show that if this differential map preserves angles, and orientation then the matrix must be a rotation dilation matrix. A good way to get started is to note that

$$\angle\, dF(\vec{v}), dF(\vec{w}) = \angle\, \vec{v}, \vec{w} \qquad \forall\, \vec{v}, \vec{w} \in T_{(x_0, y_0)}\mathbb{R}^2$$

implies that the two columns of the derivative matrix must perpendicular, by the choice $\vec{v} = [1, 0]^T, \vec{w} = [0, 1]^T$. Then make use of the dot product formula you know for (unoriented) angles, for at least one other good choice of $\vec{v}, \vec{w}$, to deduce that the magnitudes of the columns must agree.

$$\cos(\theta) = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|\ \|\vec{w}\|}.$$

Finally, use the fact that two ordered vectors $\vec{v}, \vec{w}$ are positively oriented means that the determinant of the matrix with columns $[\vec{v}\ \vec{w}]$ is positive. (Geometrically this means that the signed angle from $\vec{v}$ to $\vec{w}$ is between 0 and π.) The differential map is orientation preserving means that it transforms positively oriented vectors to positively oriented vectors. (Geometrically this means that the differnential map is not a reflectioni-dilation.) As an aside, using determininants is how you define positive orientation for n vectors in $\mathbb{R}^n$, as the right hand rule no longer makes any sense when $n > 3$.