Classic texts Despite the large numbers of texts written in recent years, some of the older classics remain the best. A few that are worth looking at are A. Hurwitz and R. Courant, *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* (Berlin: Julius Springer, 1925); E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Fourth Edition (London: Cambridge University Press, 1927); E. T. Titchmarch, *The Theory of Functions*, 2d ed. (NewYork: Oxford University Press, 1939, reprinted 1985); and K. Knopp, *Theory of Functions* (New York: Dover, 1947). The reader who wishes further information on various of the more advanced topics can profitably consult E. Hille, *Analytic Function Theory*, 2 volumes, (Boston: Ginn, 1959); L. V. Ahlfors, *Complex Analysis* (New York: McGraw-Hill, 1966); W. Rudin, *Real and Complex Analysis* (New York: McGraw-Hill, 1969); and P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (New York: McGraw-Hill, 1953). Some additional references are given throughout the text.

The modern treatment of complex analysis did not evolve rapidly or smoothly. The numerous creators of this area of mathematics traveled over many rough roads and encountered many blind alleys before the superior routes were found. An appreciation of the history of mathematics and its intimate connection to the physical sciences is important to every student's education. We recommend looking at M. Klein's *Mathematical Thought from Ancient to Modern Times* (London: Oxford University Press, 1972).

Third edition The third edition features an *Instructors Supplement* as well as a *Student Guide*. Answers to the odd numbered problems are in the back of the book and exercises with solutions in the Student Guide are marked with a bullet (•) in the text.

We have streamlined a number of features in the text, such as the treatment of Cauchy's Theorem. We have substantially rewritten Chapter 4 on the evaluation of integrals, making the treatment less encyclopedic. An *Internet Supplement* is available free from http://www.whfreeman.com/ (look in the mathematics section) or from http://cds.caltech.edu/~marsden/ (look under "books") contains additional information for those who want to delve into some topics in a little more depth.

Acknowledgments We are grateful to the many readers who supplied corrections and comments for this edition. There are too many to be thanked individually, but we would like especially to mention (more or less chronologically) M. Buchner (who helped significantly with the First Edition), C. Risk, P. Roeder, W. Barker, G. Hill, J. Seitz, J. Brudowski, H. O. Cordes, M. Choi, W. T. Stallings, E. Green, R. Iltis, N. Starr, D. Fowler, L. L. Campbell, D. Goldschmidt, T. Kato, J. Mesirov, P. Kenshaft, K. L. Teo, G. Bergmann, J. Harrison and C. Daniels. Finally, we thank Barbara Marsden for her accurate typsetting of this new edition.

Chapter 1

Analytic Functions

In this chapter the basic ideas about complex numbers and analytic functions are introduced. The organization of the text is analogous to that of an elementary calculus textbook, which begins by introducing $\mathbb{R}$, the set of real numbers, and functions $f(x)$ of a real variable x. One then studies the theory and practice of differentiation and integration of functions of a real variable. Similarly, in complex analysis we begin by introducing $\mathbb{C}$, the set of complex numbers z. We then study functions $f(z)$ of a complex variable z, which are differentiable in a complex sense; these are called analytic functions.

The analogy between real and complex variables is, however, a little deceptive, because complex analysis is a surprisingly richer theory; a lot more can be said about an analytic function than about a differentiable function of a real variable, as will be fully developed in subsequent chapters.

In addition to becoming familiar with the theory, the student should strive to gain facility with the standard (or "elementary") functions—such as polynomials, e^z, $\log z$, $\sin z$—as in calculus. These functions are studied in §1.3 and appear frequently throughout the text.

1.1 Introduction to Complex Numbers

The following discussion will assume some familiarity with the main properties of real numbers. The real number system resulted from the search for a system (an abstract set together with certain rules) that included the rationals but that also provided solutions to such polynomial equations as $x^2 - 2 = 0$.

Historical Perspective Historically, a similar consideration gave rise to an extension of the real numbers. As early as the sixteenth century, Geronimo Cardano considered quadratic (and cubic) equations such as $x^2 + 2x + 2 = 0$, which is satisfied by no real number x. The quadratic formula $\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$ yields "formal" expressions for the two solutions of the equation $ax^2 + bx + c = 0$. But this formula may involve square roots of negative numbers; for example, $-1 \pm \sqrt{-1}$ for the equation $x^2 + 2x + 2 = 0$. Cardano noticed that if these "complex numbers" were treated as ordinary numbers with the added rule that $\sqrt{-1} \cdot \sqrt{-1} = -1$, they did indeed solve the equations.

The important expression $\sqrt{-1}$ is now given the widely accepted designation $i = \sqrt{-1}$. (An alternative convention is followed by many electrical engineers, who prefer the symbol $j = \sqrt{-1}$ since they wish to reserve the symbol i for electric current.) However, in the past it was felt that no meaning could actually be assigned to such expressions, which were therefore termed "imaginary." Gradually, especially as a result of the work of Leonhard Euler in the eighteenth century, these imaginary quantities came to play an important role. For example, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ revealed the existence of a profound relationship between complex numbers and the trigonometric functions. The rule $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$ was found to summarize the rules for expanding sine and cosine of a sum of two angles in a neat way, and this result alone indicated that some meaning should be attached to these "imaginary" numbers.

However, it took nearly three hundred years until the work of Casper Wessel (ca. 1797), Jean Robert Argand (1806), Karl Friedrich Gauss (1831), Sir William R. Hamilton (1837), and others, when "imaginary" numbers were recognized as legitimate mathematical objects, and it was realized that there is nothing "imaginary" about them at all (although this term is still used).

The complex analysis that is the subject of this book was developed in the nineteenth century, mainly by Augustin Cauchy (1789–1857). Later his theory was made more rigorous and extended by such mathematicians as Peter Dirichlet (1805–1859), Karl Weierstrass (1815–1897), and Georg Friedrich Bernhard Riemann (1826–1866).

The search for a method to describe heat conduction influenced the development of the theory, which has found many uses outside mathematics. Subsequent chapters will discuss some of these applications to problems in physics and engineering, such as hydrodynamics and electrostatics. The theory also has mathematical applications to problems that at first do not seem to involve complex numbers. For example, the proof that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2},$$

or that

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin(\alpha\pi)},$$

(where $0 < \alpha < 1$), or that

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = \frac{2\pi}{\sqrt{a^2-1}},$$

may be difficult or, in some cases, impossible using elementary calculus, but these identities can be readily proved using the techniques of complex variables.

The Complex Number System Complex analysis has become an indispensable and standard tool of the working mathematician, physicist, and engineer. Neglect of it can prove to be a severe handicap in most areas of research and application involving mathematical ideas and techniques. The first objective of this section will be to define complex numbers and to show that the usual algebraic manipulations hold. To begin, recall that the xy plane, denoted by $\mathbb{R}^2$, consists of all ordered pairs (x, y) of real numbers.

Definition 1.1.1 *The **system of complex numbers,** denoted $\mathbb{C}$, is the set $\mathbb{R}^2$ together with the usual rules of **vector addition** and **scalar multiplication** by a real number a, namely,*

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ a(x, y) &= (ax, ay)\end{aligned}$$

*and with the operation of **complex multiplication,** defined by*

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

We will need to explain where this strange rule of multiplication comes from! Rather than using (x, y) to represent a complex number, we will find it more convenient to return to more standard notation as follows. Let us identify *real* numbers x with points on the x axis; thus x and $(x, 0)$ stand for the same point $(x, 0)$ in $\mathbb{R}^2$. The y axis will be called the ***imaginary axis***, and the unit point $(0, 1)$ will be denoted i. Thus, by definition, $i = (0, 1)$. Then

$$(x, y) = x + yi$$

because the right side of the equation stands for

$$(x, 0) + y(0, 1) = (x, 0) + (0, y) = (x, y).$$

Using $y = (y, 0)$ and Definition 1.1.1 of complex multiplication, we get

$$iy = (0, 1)(y, 0) = (0 \cdot y - 1 \cdot 0, y \cdot 1 + 0 \cdot 0) = (0, y) = y(0, 1) = yi,$$

so we can also write $(x, y) = x + iy$. A single symbol such as $z = a + ib$ is generally used to indicate a complex number. The notation $z \in \mathbb{C}$ means that z belongs to the set of complex numbers.

Note that

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, (1 \cdot 0 + 0 \cdot 1)) = (-1, 0) = -1,$$

so we do have the property we want:

$$i^2 = -1.$$

If we remember this equation, then the rule for multiplication of complex numbers is also easy to remember and motivate:

$$\begin{aligned}(a+ib)(c+id) &= ac+iad+ibc+i^2bd\\ &= (ac-bd)+i(ad+bc).\end{aligned}$$

For example, $2+3i$ is the complex number $(2,3)$, and

$$(2+3i)(1-4i)=2-12i^2+3i-8i=14-5i$$

is another way of saying that

$$(2,3)(1,-4)=(2\cdot 1-3(-4),3\cdot 1+2(-4))=(14,-5).$$

The reason for using the expression $a+bi$ is twofold. First, it is conventional. Second, the rule $i^2=-1$ is easier to use than the rule $(a,b)(c,d)=(ac-bd,bc+ad)$, although both rules produce the same result.

Because multiplication of real numbers is associative, commutative, and distributive, it is reasonable to expect that multiplication of complex numbers is also; that is, for all complex numbers z, w, and s we have

$$(zw)s=z(ws),\quad zw=wz,\quad\text{and}\quad z(w+s)=zw+zs.$$

Let us verify the first of these properties; the others can be similarly verified.

Let $z=a+ib, w=c+id$, and $s=e+if$. Then $zw=(ac-bd)+i(bc+ad)$, so

$$(zw)s=e(ac-bd)-f(bc+ad)+i[e(bc+ad)+f(ac-bd)].$$

Similarly,

$$\begin{aligned}z(ws) &= (a+bi)[(ce-df)+i(cf+de)]\\ &= a(ce-df)-b(cf+de)+i[a(cf+de)+b(ce-df)].\end{aligned}$$

Comparing these expressions and accepting the usual properties of real numbers, we conclude that $(zw)s=z(ws)$. Thus we can write, without ambiguity, an expression like $z^n=z\cdot\ldots\cdot z$ (n times).

Note that $a+ib=c+id$ means $a=c$ and $b=d$ (since this is what equality means in $\mathbb{R}^2$) and that 0 stands for $0+i0=(0,0)$. Thus $a+ib=0$ means that *both* $a=0$ and $b=0$.

In what sense are these complex numbers an extension of the reals? We have already said that if a is real we also write a to stand for $a+0i=(a,0)$. In other words, the reals $\mathbb{R}$ are identified with the x axis in $\mathbb{C}=\mathbb{R}^2$; we are thus regarding the real numbers as those complex numbers $a+bi$ for which $b=0$. If, in the expression $a+bi$, the term $a=0$, we call $bi=0+bi$ a ***pure imaginary number.*** In the expression $a+bi$ we say that a is the ***real part*** and b is the ***imaginary part.*** This is sometimes written $\operatorname{Re} z=a$, $\operatorname{Im} z=b$, where $z=a+bi$. Note that $\operatorname{Re} z$ and $\operatorname{Im} z$ are always real numbers (see Figure 1.1.1).

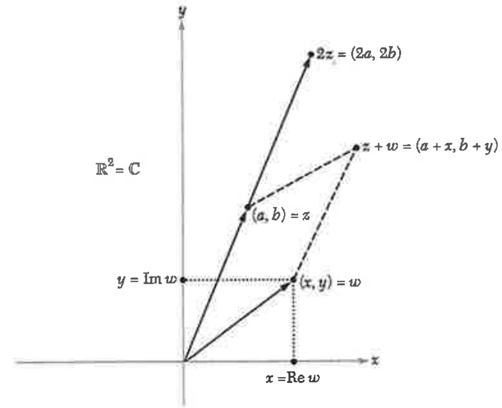

Figure 1.1.1: The geometry of complex numbers.

Algebraic Properties The complex numbers obey all the algebraic rules that ordinary real numbers do. For example, it will be shown in the following discussion that multiplicative inverses exist for nonzero elements. This means that if $z\neq 0$ then there is a (complex) number z' such that $zz'=1$, and we write $z'=z^{-1}$. We can write this expression unambiguously (in other words, z' is uniquely determined) because if $zz''=1$ as well, then $z'=z'\cdot 1=z'(zz'')=(z'z)z''=1\cdot z''=z''$, and so $z''=z'$. To show that z' exists, suppose that $z=a+ib\neq 0$. Then at least one of $a\neq 0, b\neq 0$ holds, and so $a^2+b^2\neq 0$. To find z', we set $z'=a'+b'i$. The condition $zz'=1$ imposes conditions that will enable us to compute a' and b'. Computing the product gives $zz'=(aa'-bb')+(ab'+a'b)i$. The linear equations $aa'-bb'=1$ and $ab'+a'b=0$ can be solved for a' and b' giving $a'=a/(a^2+b^2)$ and $b'=-b/(a^2+b^2)$, since $a^2+b^2\neq 0$. Thus for $z=a+ib\neq 0$, we may write

$$z^{-1}=\frac{a}{a^2+b^2}-\frac{ib}{a^2+b^2}.$$

Having found this candidate for z^{-1} it is now a straightforward, albeit tedious computation to check that it works.

If z and w are complex numbers with $w\neq 0$, then the symbol z/w means zw^{-1}; we call z/w the ***quotient*** of z by w. Thus $z^{-1}=1/z$. To compute z^{-1} the following series of equations is common and is a useful way to remember the preceding formula for z^{-1}:

$$\frac{1}{a+ib}=\frac{a-ib}{(a+ib)(a-ib)}=\frac{a-ib}{a^2+b^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i.$$

In short, all the usual algebraic rules for manipulating real numbers, fractions, polynomials, and so on, hold for complex numbers.

Formally, the system of complex numbers is an example of a ***field***. The crucial rules for a field, stated here for reference, are

Addition rules

(i) $z+w=w+z$

(ii) $z+(w+s)=(z+w)+s$

(iii) $z+0=z$

(iv) $z+(-z)=0$

Multiplication rules

(i) $zw=wz$

(ii) $(zw)s=z(ws)$

(iii) $1z=z$

(iv) $z(z^{-1})=1$ for $z\neq 0$

Distributive law $z(w+s)=zw+zs$

In summary, we have

Theorem 1.1.2 *The complex numbers $\mathbb{C}$ form a field.*

The student is cautioned that we generally do not define inequalities like $z\leq w$, for complex z and w. If one requires the usual ordering properties for reals to hold, then *such an ordering is impossible* for complex numbers.[1] Thus in this text the notation $z\leq w$ will be avoided unless z and w happen to be real.

Roots of Quadratic Equations As mentioned previously, one of the reasons for using complex numbers is to enable us to take square roots of negative real numbers. That this can, in fact, be done for all complex numbers is verified in the next proposition.

Proposition 1.1.3 *Let $z\in\mathbb{C}$. Then there exists a complex number $w\in\mathbb{C}$ such that $w^2=z$. (Notice that $-w$ also satisfies this equation.)*

[1]This statement can be proved as follows. Suppose that such an ordering exists. Then either $i\geq 0$ or $i\leq 0$. Suppose that $i\geq 0$. Then $i\cdot i\geq 0$, so $-1\geq 0$, which is absurd. Alternatively, suppose that $i\leq 0$. Then $-i\geq 0$, so $(-i)(-i)\geq 0$, that is, $-1\geq 0$, again absurd. If $z=a+ib$ and $w=c+id$, we could say that $z\leq w$ iff $a\leq c$ and $b\leq d$. This is an ordering of sorts, but it does not satisfy all the rules that might be required, such as those obeyed by real numbers.

Proof (We shall give a purely algebraic proof here; another proof, based on polar coordinates, is given in §1.2.) Let $z=a+bi$. We want to find $w=x+iy$ such that $z=w^2$; *i.e.*, $a+bi=(x+iy)^2=(x^2-y^2)+(2xy)i$, and so we must simultaneously solve $x^2-y^2=a$ and $2xy=b$. The existence of such solutions is geometrically clear from examination of the graphs of the two equations. These graphs are shown in Figure 1.1.2 for the case in which both a and b are positive. From the graphs it is clear that there should be two solutions which are negatives of each other. In the following paragraph, these will be obtained algebraically.

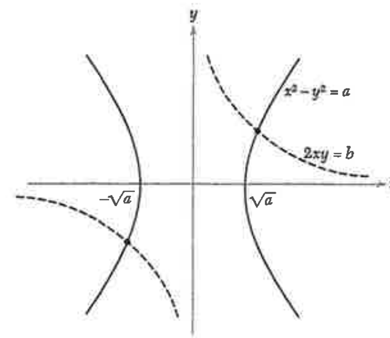

Figure 1.1.2: Graphs of the curves $x^2-y^2=a$ and $2xy=b$.

We know that $(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2=a^2+b^2$. Hence $x^2+y^2=\sqrt{a^2+b^2}$, so $x^2=\left(a+\sqrt{a^2+b^2}\right)/2$ and $y^2=\left(-a+\sqrt{a^2+b^2}\right)/2$. If we let

$$\alpha=\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}\quad\text{and}\quad\beta=\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}},$$

where $\sqrt{\ }$ denotes the positive square root of positive real numbers, then, in the event that b is positive, we have either $x=\alpha, y=\beta$ or $x=-\alpha, y=-\beta$; in the event that b is negative, we have either $x=\alpha, y=-\beta$ or $x=-\alpha, y=\beta$. We conclude that the equation $w^2=z$ has solutions $\pm(\alpha+\mu\beta i)$, where $\mu=1$ if $b\geq 0$ and $\mu=-1$ if $b<0$. ■

The formula for square roots developed in this proof is worth summarizing explicitly. Namely, *the two (complex) square roots of $a+ib$ are given by*

$$\sqrt{a+ib}=\pm(\alpha+\mu\beta i),$$

where α and β are given by the displayed formula preceding this one and where $\mu=1$ if $b\geq 0$ and $\mu=-1$ if $b<0$. From the expressions for α and β we can conclude three things:

1. The square roots of a complex number are real if and only if the complex number is real and positive.
2. The square roots of a complex number are purely imaginary if and only if the complex number is real and negative.
3. The two square roots of a number coincide if and only if the complex number is zero.

(The student should check these conclusions.)

We can easily check that the quadratic equation $az^2 + bz + c = 0$ for complex numbers a, b, c has solutions $z = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$, where now the square root denotes the two square roots just constructed.

Worked Examples

Example 1.1.4 *Prove that* $1/i = -i$ *and that* $1/(i+1) = (1-i)/2$.

Solution First,

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = -i$$

because $i \cdot -i = -(i^2) = -(-1) = 1$. Also,

$$\frac{1}{i+1} = \frac{1}{i+1}\frac{1-i}{1-i} = \frac{1-i}{2}$$

since $(1+i)(1-i) = 1+1 = 2$.

Example 1.1.5 *Find the real and imaginary parts of* $(z+2)/(z-1)$ *where* $z = x + iy$.

Solution We start by writing the fraction in terms of the real and imaginary parts of z and "rationalizing the denominator". Namely,

$$\frac{z+2}{z-1} = \frac{(x+2)+iy}{(x-1)+iy} = \frac{(x+2)+iy}{(x-1)+iy} \cdot \frac{(x-1)-iy}{(x-1)-iy} = \frac{(x+2)(x-1)+y^2+i[y(x-1)-y(x+2)]}{(x-1)^2+y^2}.$$

Hence,

$$\operatorname{Re}\frac{z+2}{z-1} = \frac{x^2+x-2+y^2}{(x-1)^2+y^2}$$

and

$$\operatorname{Im}\frac{z+2}{z-1} = \frac{-3y}{(x-1)^2+y^2}.$$

Example 1.1.6 *Solve the equation* $z^4 + i = 0$ *for* z.

Solution Let $z^2 = w$. Then the equation becomes $w^2 + i = 0$. Now we use the formula $\sqrt{a+ib} = \pm(\alpha + \mu\beta i)$ we developed for taking square roots. Letting $a = 0$ and $b = -1$, we get

$$w = \pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right).$$

Consider the equation $z^2 = (1-i)/\sqrt{2}$. Using the same formula for square roots, but now letting $a = 1/\sqrt{2}$ and $b = -1/\sqrt{2}$, we obtain the two solutions

$$z = \pm\left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2}i\right).$$

From the second possible value for w we obtain two further solutions:

$$z = \pm\left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2+\sqrt{2}}}{2}i\right).$$

In the next section, de Moivre's formula will be developed, which will enable us to find the nth root of any complex number rather simply.

Example 1.1.7 *Prove that, for complex numbers* z *and* w,

$$\operatorname{Re}(z+w) = \operatorname{Re} z + \operatorname{Re} w$$

and

$$\operatorname{Im}(z+w) = \operatorname{Im} z + \operatorname{Im} w.$$

Solution Let $z = x + iy$ and $w = a + ib$. Then $z + w = (x+a) + i(y+b)$, so $\operatorname{Re}(z+w) = x + a = \operatorname{Re} z + \operatorname{Re} w$. Similarly, $\operatorname{Im}(z+w) = y + b = \operatorname{Im} z + \operatorname{Im} w$.

Exercises

1. Express the following complex numbers in the form $a + ib$:

   (a) $(2+3i) + (4+i)$

   • (b) $\frac{2+3i}{4+i}$

   (c) $\frac{1}{i} + \frac{3}{1+i}$

2. Express the following complex numbers in the form $a + bi$:

   (a) $(2+3i)(4+i)$

   (b) $(8+6i)^2$

- (c) $\left(1+\frac{3}{1+i}\right)^2$

3. Find the solutions to $z^2 = 3 - 4i$.

4. Find the solutions to the following equations:

   - (a) $(z+1)^2 = 3+4i$
   - (b) $z^4 - i = 0$

5. Find the real and imaginary parts of the following, where $z = x + iy$:

   (a) $\frac{1}{z^2}$

   (b) $\frac{1}{3z+2}$

6. Find the real and imaginary parts of the following, where $z = x + iy$:

   - (a) $\frac{z+1}{2z-5}$

   (b) z^3

7. Is it true that $\mathrm{Re}(zw) = (\mathrm{Re}\, z)(\mathrm{Re}\, w)$?

8. If a is real and z is complex, prove that $\mathrm{Re}(az) = a\ \mathrm{Re}\, z$ and $\mathrm{Im}(az) = a\ \mathrm{Im}\, z$. Generally, show that $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ is a real linear map; that is, $\mathrm{Re}(az + bw) = a\, \mathrm{Re}\, z + b\, \mathrm{Re}\, w$ for a, b real and z, w complex.

9.• Show that $\mathrm{Re}(iz) = -\mathrm{Im}(z)$ and that $\mathrm{Im}(iz) = \mathrm{Re}(z)$ for any complex number z.

10. (a) Fix a complex number $z = x + iy$ and consider the linear mapping $\phi_z : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, of $\mathbb{C} \to \mathbb{C}$) defined by $\phi_z(w) = z \cdot w$ (that is, multiplication by z). Prove that the matrix of ϕ_z in the standard basis $(1,0), (0,1)$ of $\mathbb{R}^2$ is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

   (b) Show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.

11. Assuming that they work for real numbers, show that the nine rules given for a field also work for complex numbers. mult

12. Using only the axioms for a field, give a formal proof (including all details) for the following:

   (a) $\frac{1}{z_1 z_2} = \frac{1}{z_1} \cdot \frac{1}{z_2}$

   (b) $\frac{1}{z_1} + \frac{1}{z_2} = \frac{z_1+z_2}{z_1 z_2}$

13.• Let $(x - iy)/(x + iy) = a + ib$. Prove that $a^2 + b^2 = 1$.

14. Prove the binomial theorem for complex numbers; that is, letting z, w be complex numbers and n be a positive integer,

$$(z+w)^n = z^n + \binom{n}{1} z^{n-1} w + \binom{n}{2} z^{n-2} w^2 + \ldots + \binom{n}{n} w^n,$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Use induction on n.

15. Show that z is real if and only if $\mathrm{Re}\, z = z$.

16. Prove that, for each integer k,

$$i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i.$$

Show how this result gives a formula for i^n for all n by writing $n = 4k + j, 0 \le j \le 3$.

17. Simplify the following:

   - (a) $(1+i)^4$

   (b) $(-i)^{-1}$

18. Simplify the following:

   (a) $(1-i)^{-1}$

   (b) $\frac{1+i}{1-i}$

19. Simplify the following:

   (a) $\sqrt{1+\sqrt{i}}$

   (b) $\sqrt{1+i}$

   (c)• $\sqrt{\sqrt{-i}}$

20. Show that the following rules uniquely determine complex multiplication on $\mathbb{C} = \mathbb{R}^2$:

   (a) $(z_1 + z_2)w = z_1 w + z_2 w$

   (b) $z_1 z_2 = z_2 z_1$

   (c) $i \cdot i = -1$

   (d) $z_1(z_2 z_3) = (z_1 z_2) z_3$

   (e) If z_1 and z_2 are real, $z_1 \cdot z_2$ is the usual product of real numbers.

1.2 Properties of Complex Numbers

It is important to be able to visualize mathematical concepts and to develop geometric intuition—an ability especially valuable in complex analysis. In this section we define and give a geometric interpretation for several concepts: the *absolute value, argument, polar representation*, and *complex conjugate* of a complex number.

Addition of Complex Numbers In the preceding section a complex number was defined to be a point in the plane $\mathbb{R}^2$. Thus, a complex number may be thought of geometrically as a (two-dimensional) vector and pictured as an arrow from the origin to the point in $\mathbb{R}^2$ given by the complex number (see Figure 1.2.1).

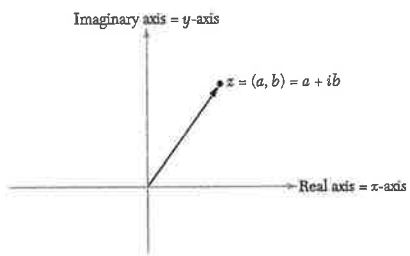

Figure 1.2.1: Vector representation of complex numbers.

Because the points $(x, 0) \in \mathbb{R}^2$ correspond to real numbers, the horizontal or x axis is called the ***real axis***. Similarly, the vertical axis (the y axis) is called the ***imaginary axis***, because points on it have the form $iy = (0, y)$ for y real.

As we already saw in Figure 1.1.1, the addition of complex numbers can be pictured as addition of vectors (an explicit example is given in Figure 1.2.2).

Polar Representation of Complex Numbers To understand the geometric meaning of multiplying two complex numbers, we will write them in what is called polar coordinate form. Recall that the ***length*** of the vector $(a, b) = a+ib$ is defined to be $\sqrt{a^2+b^2}$. Suppose the vector makes an angle θ with the positive direction of the real axis, where $0 \le \theta < 2\pi$ (see Figure 1.2.3).

Thus, $\tan\theta = b/a$. Since $a = r\cos\theta$ and $b = r\sin\theta$, we have

$$a + bi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta).$$

This way of writing the complex number is called the ***polar coordinate representation***. The length of the vector $z = (a, b) = a + ib$ is denoted $|z|$ and is called the ***norm***, or ***modulus***, or ***absolute value*** of z. The angle θ is called the ***argument*** of the complex number and is denoted $\theta = \arg z$.

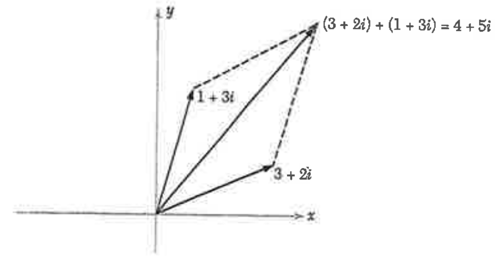

Figure 1.2.2: Addition of complex numbers.

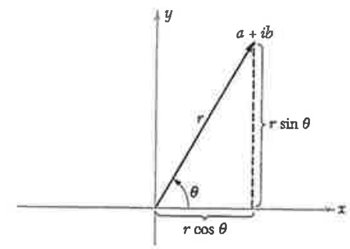

Figure 1.2.3: Polar coordinate representation of complex numbers.

If we restrict θ to the interval $0 \le \theta < 2\pi$, then each nonzero complex number has an unambiguously defined argument. (We learn this in trigonometry.) However, it is clear that we can add integral multiples of 2π to θ and still obtain the same complex number. In fact, we shall find it convenient to be flexible in our requirements for the values that θ is to assume. For example, we could equally well allow the range of θ to be $-\pi < \theta \le \pi$. Such an interval must always be specified or be clearly understood.

Once an interval of length 2π is specified, then for each $z \ne 0$, a unique θ is determined that lies within that specified interval. It is clear that any $\theta \in \mathbb{R}$ can be brought into our specified interval by the addition of some (positive or negative) integral multiple of 2π. For these reasons it is sometimes best to think of $\arg z$ as the set of possible values of the angle. If θ is one possible value, then so is $\theta + 2\pi n$ for any integer n, and we can sometimes think of $\arg z$ as $\{\theta + 2\pi n \mid n \text{ is an integer}\}$. Specification of a particular range for the angle is known as choosing a ***branch of the argument***.

Multiplication of Complex Numbers The polar representation of complex numbers helps us understand the geometric meaning of the product of two complex numbers. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2[(\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2)] + i[(\cos\theta_1 \cdot \sin\theta_2 + \cos\theta_2 \cdot \sin\theta_1)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \end{aligned}$$

by the addition formula for the sine and cosine functions used in trigonometry. Thus, we have proven

Proposition 1.2.1 *For any complex numbers z_1 and z_2,*

$$|z_1 z_2| = |z_1| \cdot |z_2| \quad \textit{and} \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}.$$

In other words, the product of two complex numbers is the complex number that has a length equal to the product of the lengths of the two complex numbers and an argument equal to the sum of the arguments of those numbers. This is the basic geometric representation of complex multiplication (see Figure 1.2.4).

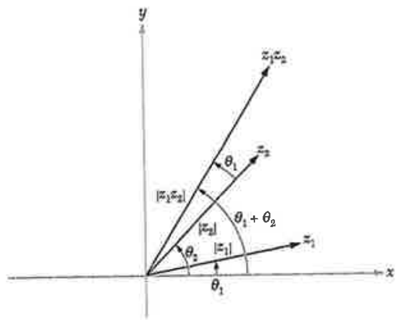

Figure 1.2.4: Multiplication of complex numbers.

The second equality in Proposition 1.2.1 means that the sets of possible values for the left and right sides are the same, that is, that the two sides can be made to agree by the addition of the appropriate multiple of 2π to one side. If a particular branch is desired and $\arg z_1 + \arg z_2$ lies outside the interval that we specify, we should adjust it by a multiple of 2π to bring it within that interval. For example, if our interval is $[0, 2\pi[$ and $z_1 = -1$ and $z_2 = -i$, then $\arg z_1 = \pi$ and $\arg z_2 = 3\pi/2$ (see Figure 1.2.5), but $z_1 z_2 = i$, so $\arg z_1 z_2 = \pi/2$, and $\arg z_1 + \arg z_2 = \pi + 3\pi/2 = 2\pi + \pi/2$. We can obtain the correct answer by subtracting 2π to bring it within the interval $[0, 2\pi[$.

Multiplication of complex numbers can be analyzed in another useful way. Let $z \in \mathbb{C}$ and define $\psi_z : \mathbb{C} \to \mathbb{C}$ by $\psi_z(w) = wz$; that is, ψ_z is the map "multiplication

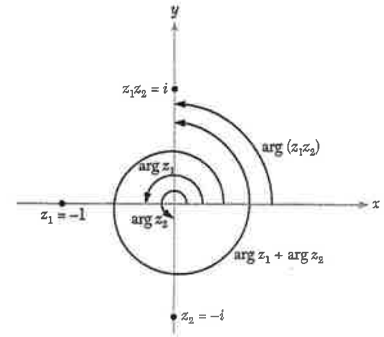

Figure 1.2.5: Multiplication of the complex numbers -1 and $-i$.

by z". By Proposition 1.2.1, *the effect of this map is to rotate a complex number through an angle equal to $\arg z$ in the counterclockwise direction and to stretch its length by the factor $|z|$.* For example, ψ_i (multiplication by i) rotates complex numbers by $\pi/2$ in the counterclockwise direction (see Figure 1.2.6).

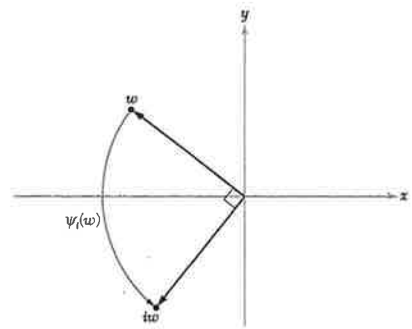

Figure 1.2.6: Multiplication by i.

The map ψ_z is a linear transformation on the plane, in the sense that $\psi_z(\lambda w_1 + \mu w_2) = \lambda\psi_z(w_1) + \mu\psi_z(w_2)$, where λ, μ are real numbers and w_1, w_2 are complex numbers. Any linear transformation of the plane to itself can be represented by a matrix, as we learn in linear algebra. If $z = a + ib = (a, b)$, then the matrix of ψ_z is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

since

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix}$$

(see Exercise 10, §1.1).

De Moivre's Formula The formula we derived for multiplication, using the polar coordinate representation, provides more than geometric intuition. We can use it to obtain a formula that enables us to find the nth roots of any complex number.

Proposition 1.2.2 (De Moivre's Formula) *If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then*

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

Proof By Proposition 1.2.1,

$$z^2 = r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)] = r^2(\cos 2\theta + i \sin 2\theta).$$

Multiplying again by z gives

$$z^3 = z \cdot z^2 = r \cdot r^2[\cos(2\theta + \theta) + i\sin(2\theta + \theta)] = r^3(\cos 3\theta + i \sin 3\theta).$$

This procedure may be continued by induction to obtain the desired result for any integer n. ■

Let w be a complex number; that is, let $w \in \mathbb{C}$. Using de Moivre's formula will help us solve the equation $z^n = w$ for z when w is given. Suppose that $w = r(\cos\theta + i\sin\theta)$ and $z = \rho(\cos\psi + i\sin\psi)$. Then de Moivre's formula gives $z^n = \rho^n(\cos n\psi + i\sin n\psi)$. It follows that $\rho^n = r = |w|$ by uniqueness of the polar representation and $n\psi = \theta + k(2\pi)$, where k is some integer. Thus

$$z = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{k}{n}2\pi\right) + i\sin\left(\frac{\theta}{n} + \frac{k}{n}2\pi\right)\right].$$

Each value of $k = 0, 1, \ldots, n-1$ gives a different value of z. Any other value of k merely repeats one of the values of z corresponding to $k = 0, 1, 2, \ldots, n-1$. Thus there are exactly n nth roots of a (nonzero) complex number.

An example will help illustrate how to use this theory. Consider the problem of finding the three solutions to the equation $z^3 = 1 = 1(\cos 0 + i \sin 0)$. The preceding formula gives them as follows:

$$z = \cos\frac{k2\pi}{3} + i\sin\frac{k2\pi}{3},$$

where $k = 0, 1, 2$. In other words, the solutions are

$$z = 1, \quad -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad -\frac{1}{2} - \frac{i\sqrt{3}}{2}.$$

This procedure forfinding roots is summarized as follows.

Corollary 1.2.3 *Let w be a nonzero complex number with polar representation $w = r(\cos\theta + i\sin\theta)$. Then the nth roots of w are given by the n complex numbers*

$$z_k = \sqrt[n]{2}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right] \quad k = 0, 1, \ldots, n-1.$$

As a special case of this formula we note that the n roots of 1 (that is, the nth roots of unity) are 1 and $n - 1$ other points equally spaced around the unit circle, as illustrated in Figure 1.2.7 for the case $n = 8$.

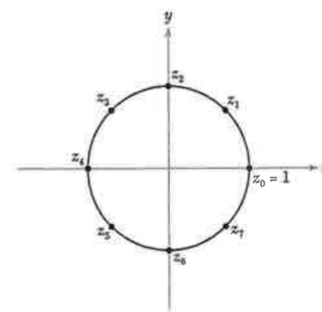

Figure 1.2.7: The eighth roots of unity.

Complex Conjugation Subsequent chapters will include many references to the simple idea of conjugation, which is defined as follows: If $z = a + ib$, then $\bar{z}$, the ***complex conjugate*** of z, is defined by $\bar{z} = a - ib$. Complex conjugation can be pictured geometrically as reflection in the real axis (see Figure 1.2.8).

The next proposition summarizes the main properties of complex conjugation.

Proposition 1.2.4 *The following properties hold for complex numbers:*

(i) $\overline{z + z'} = \bar{z} + \bar{z'}$.

(ii) $\overline{zz'} = \bar{z}\bar{z'}$.

(iii) $\overline{z/z'} = \bar{z}/\bar{z'}$ *for* $z' \neq 0$.

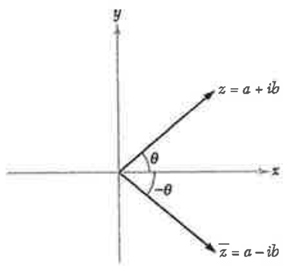

Figure 1.2.8: Complex conjugation.

(iv) $z\bar{z} = |z|^2$ *and hence if* $z \neq 0$, *we have* $z^{-1} = \bar{z}/|z|^2$.

(v) $z = \bar{z}$ *if and only if* z *is real.*

(vi) $\text{Re } z = (z + \bar{z})/2$ *and* $\text{Im } z = (z - \bar{z})/2i$.

(vii) $\bar{\bar{z}} = z$.

Proof

(i) Let $z = a + ib$ and let $z' = a' + ib'$. Then $z + z' = a + a' + i(b + b')$, and so $\overline{z + z'} = (a + a') - i(b + b') = a - ib + a' - ib' = \bar{z} + \bar{z'}$.

(ii) Let $z = a + ib$ and let $z' = a' + ib'$. Then

$$\overline{zz'} = \overline{(aa' - bb') + i(ab' + a'b)} = (aa' - bb') - i(ab' + a'b).$$

On the other hand, $\bar{z}\bar{z'} = (a - ib)(a' - ib') = (aa' - bb') - i(ab' + a'b)$.

(iii) By (ii) we have $\overline{z'}\,\overline{z/z'} = \overline{z' z/z'} = \bar{z}$. Hence, $\overline{z/z'} = \bar{z}/\bar{z'}$.

(iv) $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$.

(v) If $a + ib = a - ib$, then $ib = -ib$, and so $b = 0$.

(vi) This assertion is clear by the definition of $\bar{z}$.

(vii) This assertion is also clear by the definition of complex conjugation. ■

The absolute value of a complex number $|z| = |a + ib| = \sqrt{a^2 + b^2}$, which is the usual Euclidean length of the vector representing the complex number, has already been defined. From Proposition 1.2.4(iv), note that $|z|$ is also given by $|z|^2 = z\bar{z}$. The absolute value of a complex number is encountered throughout complex analysis; the following properties of the absolute value are quite basic.

Proposition 1.2.5 (i) $|zz'| = |z| \cdot |z'|$.

(ii) *If* $z' \neq 0$, *then* $|z/z'| = |z|/|z'|$.

(iii) $-|z| \leq \text{Re } z \leq |z|$ *and* $-|z| \leq \text{Im } z \leq |z|$; *that is,* $|\text{Re } z| \leq |z|$ *and* $|\text{Im } z| \leq |z$

(iv) $|\bar{z}| = |z|$.

(v) $|z + z'| \leq |z| + |z'|$.

(vi) $|z - z'| \geq ||z| - |z'||$.

(vii) $|z_1 w_1 + \ldots + z_n w_n| \leq \sqrt{|z_1|^2 + \ldots + |z_n|^2}\sqrt{|w_1|^2 + \ldots + |w_n|^2}$.

Statement (iv) is clear geometrically from Figure 1.2.8, (v) is called the ***trian gle inequality*** for vectors in $\mathbb{R}^2$ (see Figure 1.2.9) and (vii) is referred to as th ***Cauchy-Schwarz inequality***. By repeated application of (v) we get the genera statement $|z_1 + \ldots + z_n| \leq |z_1| + \ldots + |z_n|$.

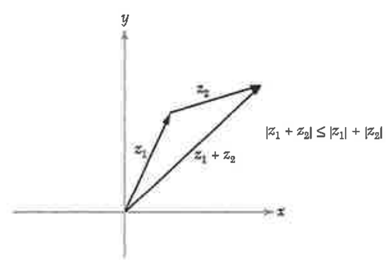

Figure 1.2.9: Triangle inequality.

Proof

(i) This equality was shown in Proposition 1.2.1.

(ii) By (i), $|z'||z/z'| = |z' \cdot (z/z')| = |z|$, so $|z/z'| = |z|/|z'|$.

(iii) If $z = a + ib$, then $-\sqrt{a^2 + b^2} \leq a \leq \sqrt{a^2 + b^2}$ since $b^2 \geq 0$. The othe inequality asserted in (iii) is similarly proved.

(iv) If $z = a + ib$, then $\bar{z} = a - ib$, and we clearly have $|z| = \sqrt{a^2 + b^2} = \sqrt{a^2 + (-b)^2} = |\bar{z}|$.

(v) By Proposition 1.2.4(iv),

$$\begin{aligned} |z+z'|^2 &= (z+z')\overline{(z+z')} \\ &= (z+z')(\bar z+\bar{z'}) \\ &= z\bar z + z'\bar{z'} + z'\bar z + z\bar{z'}. \end{aligned}$$

But $z\bar{z'}$ is the conjugate of $z'\bar z$ (Why?), so by Proposition 1.2.4(vi) and (iii) in this proof,

$$|z|^2 + |z'|^2 + 2\,\mathrm{Re}\, z'\bar z \le |z|^2 + |z'|^2 + 2|z'\bar z| = |z|^2 + |z'|^2 + 2|z||z'|.$$

But this equals $(|z| + |z'|)^2$, so we get our result.

(vi) By applying (v) to z' and $z - z'$ we get

$$|z| = |z' + (z - z')| \le |z'| + |z - z'|,$$

so $|z - z'| \ge |z| - |z'|$. By interchanging the roles of z and z', we similarly get $|z - z'| \ge |z'| - |z| = -(|z| - |z'|)$, which is what we originally claimed.

(vii) This inequality is less evident and the proof of it requires a slight mathematical trick (see Exercise 22 for a different proof). Let us suppose that not all the $w_k = 0$ (or else the result is clear). Let

$$v = \sum_{k=1}^{n} |z_k|^2 \quad t = \sum_{k=1}^{n} |w_k|^2 \quad s = \sum_{k=1}^{n} z_k w_k \quad \text{and} \quad c = s/t.$$

Now consider the sum

$$\sum_{k=1}^{n} |z_k - c\bar w_k|^2$$

which is ≥ 0 and equals

$$\begin{aligned} v + |c|^2 t - c\sum_{k=1}^{n} \bar z_k \bar w_k - \bar c \sum_{k=1}^{n} z_k w_k &= v + |c|^2 t - 2\,\mathrm{Re}\,\bar c s \\ &= v + \frac{|s|^2}{t} - 2\,\mathrm{Re}\,\frac{\bar s s}{t}. \end{aligned}$$

Since t is real and $s\bar s = |s|^2$ is real, $v + (|s|^2/t) - 2(|s|^2/t) = v - |s|^2/t \ge 0$. Hence $|s|^2 \le vt$, which is the desired result. ■

Worked Examples

Example 1.2.6 *Solve $z^8 = 1$ for z.*

Solution Since $1 = \cos k2\pi + i \sin k2\pi$ when k equals any integer, Corollary 1 gives

$$\begin{aligned} z &= \cos\frac{k2\pi}{8} + i\sin\frac{k2\pi}{8} \quad k = 0,1,2,\dots,7 \\ &= 1, \frac{1}{\sqrt2} + \frac{i}{\sqrt2}, i, \frac{-1}{\sqrt2} + \frac{i}{\sqrt2}, -1, \frac{-1}{\sqrt2} - \frac{i}{\sqrt2}, -i, \frac{1}{\sqrt2} - \frac{i}{\sqrt2}. \end{aligned}$$

These may be pictured as points evenly spaced on the circle in the complex pl (see Figure 1.2.10).

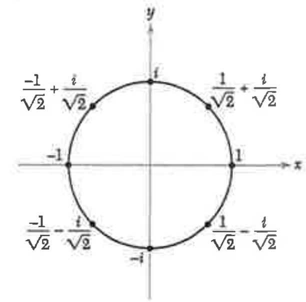

Figure 1.2.10: The eight 8th roots of unity.

Example 1.2.7 *Show that*

$$\overline{\left[\frac{(3+7i)^2}{(8+6i)}\right]} = \frac{(3-7i)^2}{(8-6i)}.$$

Solution The point here is that it is not necessary first to work out $(3+7i)^2/(8+6i)$ if we simply use the properties of complex conjugation, namely, $\overline{z^2} = (\bar z)^2$ and $\overline{z/z'} = \bar z/\bar{z'}$. Thus we obtain

$$\overline{\left[\frac{(3+7i)^2}{(8+6i)}\right]} = \frac{\overline{(3+7i)^2}}{\overline{(8+6i)}} = \frac{\left(\overline{3+7i}\right)^2}{\overline{(8+6i)}} = \frac{(3-7i)^2}{(8-6i)}.$$

Example 1.2.8 *Show that the maximum absolute value of z^2+1 on the unit disk $|z| \le 1$ is 2.*

Solution By the triangle inequality, $|z^2+1| \le |z^2| + 1 = |z|^2 + 1 \le 1^2 + 1 = 2$ since $|z| \le 1$ thus $|z^2+1|$ does not exceed 2 on the disk. Since the value 2 is achieved at $z = 1$, the maximum is 2.

Example 1.2.9 *Express* $\cos 3\theta$ *in terms of* $\cos\theta$ *and* $\sin\theta$ *using de Moivre's formula.*

Solution De Moivre's formula for $r = 1$ and $n = 3$ gives the identity

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

The left side of this equation, when expanded (see Exercise 14 of §1.1), becomes

$$\cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

By equating real and imaginary parts, we get

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and the additional formula

$$\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta.$$

Example 1.2.10 *Write the equation of a straight line, of a circle, and of an ellipse using complex notation.*

Solution The straight line is most conveniently expressed in parametric form: $z = a + bt, a, b \in \mathbb{C}, t \in \mathbb{R}$, which represents a line in the direction of b and passing through the point a.

The circle can be expressed as $|z - a| = r$ (radius r, center a).

The ellipse can be expressed as $|z - d| + |z + d| = 2a$; the foci are located at $\pm d$ and the semimajor axis equals a.

These equations, in which $|\cdot|$ is interpreted as length, coincide with the geometric definitions of these loci.

Exercises

1. Solve the following equations:
   - (a) $z^5 - 2 = 0$
   - (b) $z^4 + i = 0$
2. Solve the following equations:
   - (a) $z^6 + 8 = 0$
   - (b) $z^3 - 4 = 0$
3. What is the complex conjugate of $(3 + 8i)^4/(1 + i)^{10}$?
4. What is the complex conjugate of $(8 - 2i)^{10}/(4 + 6i)^5$?
5. Express $\cos 5x$ and $\sin 5x$ in terms of $\cos x$ and $\sin x$.

6. Express $\cos 6x$ and $\sin 6x$ in terms of $\cos x$ and $\sin x$.
7. Find the absolute value of $[i(2 + 3i)(5 - 2i)]/(-2 - i)$.
8. Find the absolute value of $(2 - 3i)^2/(8 + 6i)^2$.
9. Let w be an nth root of unity, $w \neq 1$. Show that $1 + w + w^2 + \ldots + w^{n-1} = 0$.
10. Show that the roots of a polynomial with real coefficients occur in conjugate pairs.
11. If $a, b \in \mathbb{C}$, prove the ***parallelogram identity***: $|a-b|^2 + |a+b|^2 = 2(|a|^2 + |b|^2)$.
12. Interpret the identity in Exercise 11 geometrically.
13. When does equality hold in the triangle inequality $|z_1 + z_2 + \ldots + z_n| \le |z_1| + |z_2| + \ldots + |z_n|$? Interpret your result geometrically.
14. Assuming either $|z| = 1$ or $|w| = 1$ and $\bar{z}w \neq 1$, prove that
$$\left|\frac{z - w}{1 - \bar{z}w}\right| = 1.$$
15. Does $z^2 = |z|^2$? If so, prove this equality. If not, for what z is it true?
16. Letting $z = x + iy$, prove that $|x| + |y| \le \sqrt{2}|z|$.
17. Let $z = a + ib$ and $z' = a' + ib'$. Prove that $|zz'| = |z||z'|$ by evaluating each side.
18. Prove the following:
    - (a) $\arg\bar{z} = -\arg z (\mathrm{mod}\ 2\pi)$
    - (b) $\arg(z/w) = \arg z - \arg w (\mathrm{mod}\ 2\pi)$
    - (c) $|z| = 0$ if and only if $z = 0$
19. What is the equation of the circle with radius 3 and center $8 + 5i$ in complex notation?
20. Using the formula $z^{-1} = \bar{z}/|z|^2$, show how to construct z^{-1} geometrically.
21. Describe the set of all z such that $\mathrm{Im}\,(z + 5) = 0$.
22. Prove ***Lagrange's identity***:
$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) - \sum_{k<j} |z_k \bar{w}_j - z_j \bar{w}_k|^2.$$
Deduce the Cauchy-Schwarz inequality from your proof.

23.• Given $a \in \mathbb{C}$, find the maximum of $|z^n + a|$ for those z with $|z| \le 1$.

24. Compute the least upper bound (that is, supremum) of the set of all real numbers of the form Re (iz^3+1) such that $|z| < 2$.

25.• Prove ***Lagrange's trigonometric identity***:

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}.$$

(Assume that $\sin(\theta/2) \neq 0$.)

26. Suppose that the complex numbers z_1, z_2, z_3 satisfy the equation

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{z_1 - z_3}{z_2 - z_3}.$$

Prove that $|z_2 - z_1| = |z_3 - z_1| = |z_2 - z_3|$. *Hint*: Argue geometrically, interpreting the meaning of each statement.

27. Give a necessary and sufficient condition for

(a) z_1, z_2, z_3 to lie on a straight line.

(b) z_1, z_2, z_3, z_4 to lie on a straight line or a circle.

28. Prove the identity

$$\left(\sin\frac{\pi}{n}\right)\left(\sin\frac{2\pi}{n}\right)\cdots\left(\sin\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

Hint: The given product can be written as $1/2^{n-1}$ times the product of the nonzero roots of the polynomial $(1-z)^n - 1$.

29. Let w be an nth root of unity, $w \neq 1$. Evaluate $1 + 2w + 3w^2 + \ldots + nw^{n-1}$.

30. Show that the correspondence of the complex number $z = a + bi$ with the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \psi_z$ noted in the text preceding Proposition 1.2.2 has the following properties:

(a) $\psi_{zw} = \psi_z \psi_w$.

(b) $\psi_{z+w} = \psi_z + \psi_w$.

(c) $\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) $\lambda\psi_z = \psi_{\lambda z}$ if λ is real.

(e) $\psi_{\bar{z}} = (\psi_z)^t$ (the transposed matrix).

(f) $\psi_{1/z} = (\psi_z)^{-1}$.

(g) z is real if and only if $\psi_z = (\psi_z)^t$.

(h) $|z| = 1$ if and only if ψ_z is an orthogonal matrix.

1.3 Some Elementary Functions

We learn about the trigonometric functions sine and cosine, as well as the expon tial function and the logarithmic function in elementary calculus. Recall that trigonometric functions may be defined in terms of the ratios of sides of a rig angled triangle. The definition of "angle" may be extended to include any r value, and thus $\cos\theta$ and $\sin\theta$ become real-valued functions of the real variable It is a basic fact that $\cos\theta$ and $\sin\theta$ are differentiable, with derivatives given $d(\cos\theta)/d\theta = -\sin\theta$ and $d(\sin\theta)/d\theta = \cos\theta$. Alternatively, $\cos\theta$ and $\sin\theta$ can defined by their power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.$$

The proof of convergence of these series can be found in Chapter 3 and in ma calculus texts.[2] Alternatively, $\sin x$ can be defined as the unique solution $f(x)$ the differential equation $f''(x) + f(x) = 0$ satisfying $f(0) = 0, f'(0) = 1$; and co can be defined as the unique solution to $f''(x) + f(x) = 0, f(0) = 1, f'(0) =$ (again, see a calculus text for proofs).

Exponential Function The exponential function, denoted e^x, may be defin as the unique solution to the differential equation $f'(x) = f(x)$, subject to t initial condition that $f(0) = 1$; one has to show that a unique solution exists. T exponential function can also be defined by its power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

We accept from calculus the fact that e^x is a positive, strictly increasing functic of x. Therefore, for $y > 0$, $\log y$ can be defined as the inverse function of e^x; th is, $e^{\log y} = y$. Another approach that is often used in calculus books is to *begin* t defining

$$\log y = \int_1^y \frac{1}{t}dt$$

for $y > 0$ and then to define e^x as the inverse function of $\log y$. (Many calcul books write $\ln y$ for the logarithm to the base e. As in most advanced mathematic throughout this book we will write $\log y$ for $\ln y$.

In this section these functions will be extended to the complex plane. In oth words, the functions $\sin z, \cos z, e^z$, and $\log z$ will be defined for complex z, an their restrictions to the real line will be the usual $\sin x, \cos x, e^x$, and $\log x$. Th

[2] An example is J. Marsden and A. Weinstein, *Calculus*, Second Edition (New York: Springe Verlag, 1985), Chapter 12.

extension to complex numbers should be natural in the sense that many of the familiar properties of sin, cos, exp, and log are retained.

We first extend the exponential function. We know from calculus that for real x, e^x can be represented by its Maclaurin series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

Thus, it would be reasonable to define e^{iy} by

$$1 + \frac{(iy)}{1!} + \frac{(iy)^2}{2!} + \dots$$

for $y \in \mathbb{R}$. Of course, this definition is not quite legitimate, as convergence of series in $\mathbb{C}$ has not yet been discussed. Chapter 3 will show that this series does indeed represent a well-defined complex number for each y, but for the moment the series is used *informally* as the basis for the definition that follows, which will be precise. A slight rearrangement of the series (using Exercise 16, §1.1) shows that

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right),$$

which we recognize as being $\cos y + i \sin y$. Thus we *define*

$$e^{iy} = \cos y + i \sin y.$$

So far, we have defined e^z for z along both the real and imaginary axes. How do we define $e^z = e^{x+iy}$? We desire our extension of the exponential to retain the familiar properties, and among these is the law of exponents: $e^{a+b} = e^a \cdot e^b$. This requirement forces us to define $e^{x+iy} = e^x \cdot e^{iy}$. This can be stated in a formal definition.

Definition 1.3.1 *If $z = x + iy$, then e^z is defined by $e^x(\cos y + i \sin y)$.*

Note that if z is real (that is, if $y = 0$), this definition agrees with the usual exponential function e^x. The student is cautioned that we are not, at this stage, fully justified in thinking of e^z as "e raised to the "power" of z," since we have not yet, for example, established laws of exponents for complex numbers.

There is another, again purely formal, reason for defining $e^{iy} = \cos y + i \sin y$. If we write $e^{iy} = f(y) + ig(y)$, we note that since we want $e^0 = 1$, we should have $f(0) = 1$, and $g(0) = 0$. If the exponential function is to have the familiar differentiation properties, we will need

$$ie^{iy} = f'(y) + ig'(y),$$

so when $y = 0$ we get $f'(0) = 0, g'(0) = 1$. Differentiating again gives us

$$-e^{iy} = f''(y) + ig''(y).$$

Comparing this equation with $e^{iy} = f(y) + ig(y)$, we conclude that $f''(y) + f(y) = 0, f(0) = 1$, and $f'(0) = 0$. Therefore, $f(y) = \cos y$ by the definition of $\cos y$ in terms of differential equations. Similarly, we find that $g''(y) + g(y) = 0, g(0) = 0, g'(0) = 1$ and hence $g(y) = \sin y$. Thus, we would obtain $e^{iy} = \cos y + i \sin y$ as in Definition 1.3.1.

Some of the important properties of e^z are summarized in the following proposition. To state it, we recall the definition of a periodic function. A function $f : \mathbb{C} \to \mathbb{C}$ is called ***periodic*** if there exists a $w \in \mathbb{C}$ (called a ***period***) such that $f(z + w) = f(z)$ for all $z \in \mathbb{C}$.

Proposition 1.3.2

(i) *$e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.*

(ii) *e^z is never zero.*

(iii) *If x is real, then $e^x > 1$ when $x > 0$ and $0 < e^x < 1$ when $x < 0$.*

(iv) *$|e^{x+iy}| = e^x$.*

(v) *$e^{\pi i/2} = i, e^{\pi i} = -1, e^{3\pi i/2} = -i, e^{2\pi i} = 1$.*

(vi) *e^z is periodic; each period for e^z has the form $2\pi n i$, for some integer n.*

(vii) *$e^z = 1$ iff $z = 2n\pi i$ for some integer n (positive, negative, or zero).*

Proof

(i) Let $z = x + iy$, and let $w = s + it$. By our definition of e^z,

$$\begin{aligned} e^{z+w} &= e^{(x+s)+i(y+t)} \\ &= e^{x+s}[\cos(y+t) + i\sin(y+t)] \\ &= e^x e^s[(\cos y \cos t - \sin y \sin t) + i(\sin y \cos t + \cos y \sin t)] \\ &= [e^x(\cos y + i \sin y)][e^s(\cos t + i \sin t)] \end{aligned}$$

using the addition formulas for sine and cosine and the property $e^{x+s} = e^x \cdot e^s$ for real numbers x and s. Thus $e^{z+w} = e^z \cdot e^w$ for all complex numbers z and w.

(ii) For any z, we have $e^z \cdot e^{-z} = e^0 = 1$ since we know that the usual exponential satisfies $e^0 = 1$. Thus e^z can never be zero, because if it were, then $e^z \cdot e^{-z}$ would be zero, which is not true.

(iii) We may accept this from calculus. For example,[3] obviously

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots > 1 \quad \text{when} \quad x > 0.$$

[3] Another proof utilizing the definition of e^x in terms of differential equations is as follows. Recall that e^x is the unique solution to $f'(x) = f(x)$ with $e^0 = 1$ (x real). Since e^x is continuous and is never zero, it must be strictly positive. Hence $(e^x)' = e^x$ is always positive and consequently e^x is strictly increasing. Thus for $x > 0, e^x > 1$. Similarly, for $x < 0$, we have $e^x < 1$.

(iv) Using $|zz'| = |z||z'|$ (see Proposition 1.2.5) and the facts that $e^x > 0$ and $\cos^2 y + \sin^2 y = 1$, we get

$$\begin{aligned} |e^{x+iy}| &= |e^x e^{iy}| = |e^x||e^{iy}| \\ &= e^x |\cos y + i \sin y| = e^x. \end{aligned}$$

(v) By definition, $e^{\pi i/2} = \cos(\pi/2) + i\sin(\pi/2) = i$. The proofs of the other formulas are similar.

(vi) Suppose that $e^{z+w} = e^z$ for all $z \in \mathbb{C}$. Setting $z = 0$, we get $e^w = 1$. If $w = s + ti$, then, using (iv), $e^w = 1$ implies that $e^s = 1$, so $s = 0$. Hence any period is of the form $ti, t \in \mathbb{R}$. Suppose that $e^{ti} = 1$, that is, that $\cos t + i \sin t = 1$. Then $\cos t = 1, \sin t = 0$; thus, $t = 2\pi n$ for some integer n.

(vii) $e^0 = 1$, as we have seen, and $e^{2\pi n i} = 1$ because e^z is periodic, by (vi). Conversely, $e^z = 1$ implies that $e^{z+z'} = e^{z'}$ for all z'; so by (vi), $z = 2\pi n i$ for some integer n. ■

How can we picture e^{iy}? Since $e^{iy} = (\cos y, \sin y)$, it moves along the unit circle in a counterclockwise direction as y goes from 0 to 2π. It reaches i at $y = \pi/2, -1$ at $\pi, -1$ at $3\pi/2$, and 1 again at 2π. Thus, e^{iy} is the point on the unit circle with argument y (see Figure 1.3.1).

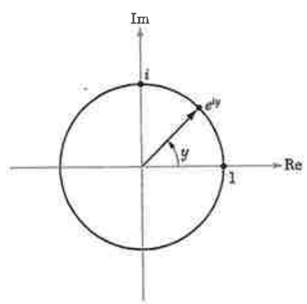

Figure 1.3.1: Points on the unit circle.

Note that in exponential form, the polar representation of a complex number becomes

$$z = |z| e^{i(\arg z)}$$

which is sometimes abbreviated to $z = re^{i\theta}$.

Trigonometric Functions Next we wish to extend the definitions of cosine a sine to the complex plane. The extension of the exponential to the complex pl suggests a way to extend the definitions of sine and cosine. We have $e^{iy} = \cos y$ $i \sin y$, and $e^{-iy} = \cos y - i \sin y$, which implies that

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \quad \text{and} \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}.$$

But since e^{iz} is now defined for any $z \in \mathbb{C}$, we are led to formulate the followi definition.

Definition 1.3.3 *The complex sine and cosine functions are defined by*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \textit{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

for any complex number z.

Again, if z is real, these definitions agree with the usual definitions of sine a cosine learned in elementary calculus.

The next proposition lists some of the properties of the sine and cosine functic that have now been defined over the whole of $\mathbb{C}$ and not merely on $\mathbb{R}$.

Proposition 1.3.4

(i) $\sin^2 z + \cos^2 z = 1$.

(ii) $\sin(z + w) = \sin z \cdot \cos w + \cos z \cdot \sin w$ *and* $\cos(z + w) = \cos z \cdot \cos w - \sin z \cdot \sin w$.

Again the student is cautioned that these formulas, although plausible, must proved, since at this stage we know their validity only when w and z are real.

Proof Using the definitions, we have

$$\begin{aligned} \sin^2 z + \cos^2 z &= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 \\ &= \frac{e^{2iz} - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} = 1, \end{aligned}$$

which proves (i). To prove (ii), write

$$\begin{aligned} &\sin z \cdot \cos w + \cos z \cdot \sin w \\ &\quad = \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iw} + e^{-iw}}{2} + \frac{e^{iz} + e^{-iz}}{2} \cdot \frac{e^{iw} - e^{-iw}}{2i}, \end{aligned}$$

which, using $e^{iz}e^{iw} = e^{i(z+w)}$ and noting cancellations between the two terr simplifies to

$$\frac{e^{i(z+w)} - e^{-i(z+w)}}{4i} + \frac{e^{i(z+w)} - e^{-i(z+w)}}{4i} = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z + w).$$

The student can similarly check the addition formula for $\cos(z+w)$. ■

In addition to $\cos z$ and $\sin z$, we can define $\tan z = (\sin z)/(\cos z)$ when $\cos z \neq 0$, and similarly obtain the other trigonometric functions.

Logarithm Function We now define the logarithm in a way that agrees with the usual definition of $\log x$ when x is real and positive. In the real case we can view the logarithm as the inverse of the exponential (that is, $\log x = y$ is the solution of $e^y = x$). When we allow z to range over $\mathbb{C}$, we must be more careful, because the exponential is periodic and thus cannot have a unique inverse. Furthermore, the exponential is never zero, so we cannot expect to be able to define the logarithm at zero. Thus, we must be careful in our choice of the domain in $\mathbb{C}$ on which we can define the logarithm. The next proposition indicates how this may be done.

Proposition 1.3.5 *Let A_{y_0} denote the set of complex numbers $x+iy$ such that $y_0 \le y < y_0 + 2\pi$; symbolically,*

$$A_{y_0} = \{x+iy \mid x \in \mathbb{R} \quad \text{and} \quad y_0 \le y < y_0 + 2\pi\}.$$

Then e^z maps A_{y_0} in a one-to-one manner onto the set $\mathbb{C}\backslash\{0\}$.

Recall that a map is ***one-to-one*** when the map takes every two distinct points to two distinct points; in other words, two distinct points never get mapped to the same point. A map is ***onto*** a set B when every point of B is the image of some point under the mapping. The notation $\mathbb{C}\backslash\{0\}$ means the whole plane $\mathbb{C}$ minus the point 0; that is, the plane with the origin removed.

Proof If $e^{z_1} = e^{z_2}$, then $e^{z_1 - z_2} = 1$, so $z_1 - z_2 = 2\pi i n$ for some integer n, by Proposition 1.3.2. But because z_1 and z_2 both lie in A_{y_0}, where the difference between the imaginary parts of any points is less than 2π, we must have $z_1 = z_2$. This argument shows that e^z is one-to-one. Let $w \in \mathbb{C}$ with $w \neq 0$. We claim the equation $e^z = w$ has a solution z in A_{y_0}. The equation $e^{x+iy} = w$ is equivalent to the two equations $e^x = |w|$ and $e^{iy} = w/|w|$. (Why?) The solution of the first equation is $x = \log|w|$, where "log" is the ordinary logarithm (with base e) defined on the positive part of the real axis. The second equation has infinitely many solutions y, each differing by integral multiples of 2π, but exactly one of these is the interval $[y_0, y_0 + 2\pi[$. This y is merely $\arg w$, where the specified range for the arg function is $[y_0, y_0+2\pi[$. Thus e^z is onto $\mathbb{C}\backslash\{0\}$. ■

The sets defined in this proposition are shown in Figure 1.3.2. Here e^z maps the horizontal strip between $y_0 i$ and $(y_0 + 2\pi)i$ one-to-one onto $\mathbb{C}\backslash\{0\}$. (The notation $z \mapsto f(z)$ is used to indicate that z is sent to $f(z)$ under the mapping f.)

In the proof of Proposition 1.3.5 an explicit expression was derived for the inverse of e^z restricted to the strip $y_0 \le \operatorname{Im} z < y_0 + 2\pi$, and this expression is stated formally in the following definition.

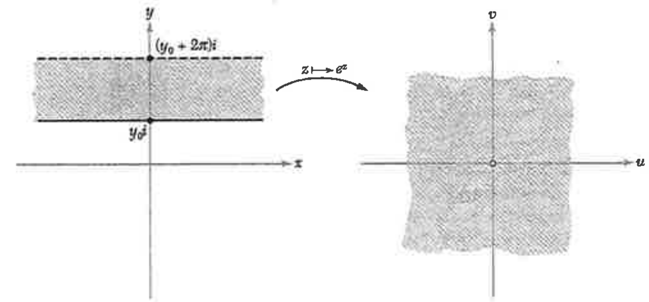

Figure 1.3.2: e^z as a one-to-one function onto $\mathbb{C}\backslash\{0\}$.

Definition 1.3.6 *The function* $\log : \mathbb{C}\backslash\{0\} \to \mathbb{C}$, *with range* $y_0 \le \operatorname{Im}\log z < y_0 + 2\pi$, *is defined by*

$$\log z = \log|z| + i \arg z,$$

where $\arg z$ *takes values in the interval* $[y_0, y_0+2\pi[$ *and* $\log|z|$ *is the usual logarithm of the positive real number* $|z|$.

This function is sometimes referred to as the "***branch*** of the logarithm function lying in $\{x+iy \mid y_0 \le y < y_0 + 2\pi\}$." But we must remember that the function $\log z$ is well defined only when we specify an interval of length 2π in which $\arg z$ takes its values, that is, when a specific branch is chosen.

For example, suppose that the specified interval for the argument is $[0, 2\pi[$. Then $\log(1+i) = \log\sqrt{2} + i\pi/4$. However, if the specified interval is $[\pi, 3\pi[$, then $\log(1+i) = \log\sqrt{2} + i9\pi/4$. Any particular branch of the logarithm defined in this way undergoes a sudden jump as z moves across the ray $\arg z = y_0$. To avoid this jumping, one can restrict the domain to $y_0 < y < y_0 + 2\pi$. This idea will be important in §1.6.

Proposition 1.3.7 *The logarithm $\log z$ is the inverse of e^z in the following sense: For any branch of $\log z$, we have $e^{\log z} = z$, and if we choose the branch lying in $y_0 \le y < y_0 + 2\pi$, then $\log(e^z) = z$ for $z = x+iy$ and $y_0 \le y < y_0 + 2\pi$.*

Proof Since $\log z = \log|z| + i\arg z$, we have

$$e^{\log z} = e^{\log|z|} e^{i \arg z} = |z| e^{i \arg z} = z.$$

Conversely, suppose that $z = x+iy$ and $y_0 \le y < y_0 + 2\pi$. By definition, $\log e^z = \log|e^z| + i \arg e^z$. But $|e^z| = e^x$ and $\arg e^z = y$ by our choice of branch. Thus, $\log e^z = \log e^x + iy = x + iy = z$. ■

The logarithm defined on $\mathbb{C}\backslash\{0\}$ behaves the same way with respect to products as the logarithm restricted to the positive part of the real axis.

Proposition 1.3.8 *If $z_1, z_2 \in \mathbb{C}\backslash\{0\}$, then $\log(z_1z_2) = \log z_1 + \log z_2$ (up to the addition of integral multiples of $2\pi i$).*

Proof By definition, $\log z_1z_2 = \log|z_1z_2| + i\arg(z_1z_2)$, where an interval $[y_0, y_0 + 2\pi[$ has been chosen for the values of the arg function. We know that $\log|z_1z_2| = \log|z_1||z_2| = \log|z_1| + \log|z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$ (up to integral multiples of 2π). Thus $\log z_1z_2 = (\log|z_1| + i\arg z_1) + (\log|z_2| + i\arg z_2) = \log z_1 + \log z_2$ (up to integral multiples of $2\pi i$). ■

To illustrate this proposition, let us find $\log[(-1-i)(1-i)]$, where the range for the arg function is chosen as, for instance, $[0, 2\pi[$. Thus,

$$\log[(-1-i)(1-i)] = \log(-2) = \log 2 + \pi i.$$

On the other hand, $\log(-1-i) = \log\sqrt{2} + i5\pi/4$ and $\log(1-i) = \log\sqrt{2} + i7\pi/4$. Thus,

$$\log(-1-i) + \log(1-i) = \log 2 + i3\pi = (\log 2 + \pi i) + 2\pi i,$$

so in this case, when $z_1 = -1-i$ and $z_2 = 1-i$, $\log z_1z_2$ differs from $\log z_1 + \log z_2$ by $2\pi i$.

The basic property in Proposition 1.3.8 can help one remember the definition of $\log z$ by writing $\log z = \log(re^{i\theta}) = \log r + \log e^{i\theta} = \log|z| + i\arg z$.

Complex Powers We are now in a position to define the expression a^b where $a, b \in \mathbb{C}$ and $a \neq 0$ (read "a raised to the power of b"). Of course, however we define a^b, the definition should reduce to the usual one in which a and b are real numbers. Notice that a can also be written $e^{\log a}$ by Proposition 1.3.7. Thus, if b is an integer, we have $a^b = (e^{\log a})^b = e^{b\log a}$. This last equality holds since if n is an integer and z is any complex number, $(e^z)^n = e^z \dots e^z = e^{nz}$ by Proposition 1.3.2(i). Thus we are led to formulate the following definition.

Definition 1.3.9 *Let $a, b \in \mathbb{C}$ with $a \neq 0$. Then a^b is defined to be $e^{b\log a}$; it is understood that some interval $[y_0, y_0 + 2\pi[$ (that is, some branch of log) has been chosen within which the arg function takes its values.*

It is important to understand precisely what this definition involves. Note especially that in general $\log z$ is "multiple-valued"; that is, $\log z$ can be assigned many different values because different intervals $[y_0 + 2\pi[$ can be chosen. This is not surprising, for if $b = 1/q$, where q is an integer, then our previous work with de Moivre's formula would lead us to expect that a^b is one of the qth roots of a and thus should have q distinct values. The following theorem elucidates this point.

Proposition 1.3.10 *Let $a, b \in \mathbb{C}, a \neq 0$. Then a^b is single-valued (that is, the value of a^b does not depend on the choice of branch for log) if and only if b is an integer. If b is a real, rational number, and if $b = p/q$ is in its lowest terms (in other words, if p and q have no common factor), then a^b has exactly q distinct values, namely, the q roots of a^p. If b is real and irrational or if b has a nonzero imaginary part, then a^b has infinitely many values. When a^b has distinct values, these values differ by factors of the form $e^{2\pi nbi}$.*

Proof Choose some interval, for example, $[0, 2\pi[$, for the values of the arg function. Let $\log z$ be the corresponding branch of the logarithm. If we were to choose any other branch of the log function, we would obtain $\log a + 2\pi ni$ rather than $\log a$, for some integer n. Thus $a^b = e^{b\log a + 2\pi nbi} = e^{b\log a} \cdot e^{2\pi nbi}$, where the value of n depends on the branch of logarithm (that is, on the interval chosen for the values of the arg function). By Proposition 1.3.2, $e^{2\pi nbi}$ remains the same for different values of n if and only if b is an integer. Similarly, $e^{2\pi nip/q}$ has q distinct values if p and q have no common factor. If b is irrational, and if $e^{2\pi nbi} = e^{2\pi mbi}$, it follows that $e^{(2\pi bi)(n-m)} = 1$ and hence $b(n-m)$ is an integer; since b is irrational, this implies that $n - m = 0$. Thus if b is irrational, $e^{2\pi nbi}$ has infinitely many distinct values. If b is of the form $x + iy, y \neq 0$, then $e^{2\pi nbi} = e^{-2\pi ny} \cdot e^{2\pi nix}$, which also has infinitely many distinct values. ■

To repeat: When we write $e^{b\log a}$, it is understood that some branch of log has been chosen, and accordingly $e^{b\log a}$ has a single well-defined value. But as we change the branch of log, we get values for $e^{b\log a}$ that differ by factors of $e^{2\pi inb}$. This is what we mean when we say that $a^b = e^{b\log a}$ is "multiple-valued".

An example should make this clear. Let $a = 1+i$ and let b be some real irrational number. Then the infinitely many different possible values of a^b are given by

$$(1+i)^b = e^{b[\log(1+i)+2\pi ni]} = e^{b(\log\sqrt{2}+i\pi/4+2\pi ni)} = (e^{b\log\sqrt{2}+ib\pi/4})e^{b2\pi ni}$$

as n takes on all integral values (corresponding to different choices of the branch). For instance, if we used the branch corresponding to $[-\pi, \pi[$ or $[0, 2\pi[$, we would set $n = 0$.

Some general properties of a^b are found in the exercises at the end of this section, but we are now interested in the special case when b is of the form $1/n$, because this gives the nth root.

The nth Root Function We know that $\sqrt[n]{z}$ has exactly n values for $z \neq 0$. To make it a specific function we single out a branch of log as described in the preceding paragraphs.

Definition 1.3.11 *The nth root function is defined by*

$$\sqrt[n]{z} = z^{1/n} = e^{(\log z)/n}$$

for a specific choice of branch of $\log z$; with this choice, $\sqrt[n]{z} = e^{(\log z)/n}$ is called a branch of the nth root function.

The next proposition verifies a familiar property of root functions.

Proposition 1.3.12 *The function* $\sqrt[n]{z}$ *so defined is an nth root of z; that is,* $(\sqrt[n]{z})^n = z$. *It is obtained as follows. If* $z = re^{i\theta}$, *then*

$$\sqrt[n]{z} = \sqrt[n]{r}e^{i\theta/n},$$

where θ is chosen so that it lies within a particular interval corresponding to the branch choice. As we add multiples of 2π to θ, we run through the n nth roots of z. On the right-hand side, $\sqrt[n]{r}$ *is the usual positive real nth root of the positive real number r.*

Proof By definition, $\sqrt[n]{z} = e^{(\log z)/n}$. But $\log z = \log r + i\theta$, so

$$e^{(\log z)/n} = e^{(\log r)/n} \cdot e^{i\theta/n} = \sqrt[n]{r}e^{i\theta/n}.$$

The assertion is then clear. ■

The reader should now take the time to become convinced that this way of describing the n nth roots of z is the same as that described in Corollary 1.2.3.

Geometry of the Elementary Functions To further understand the functions z^n, $\sqrt[n]{z}$, e^z, and $\log z$, we shall consider the geometric interpretation of each in the remainder of this section. Let us begin with the power function z^n and let $n = 2$. We know that z^2 has length $|z|^2$ and argument $2 \arg z$. Thus the map $z \mapsto z^2$ *squares lengths* and *doubles arguments* (see Figure 1.3.3).

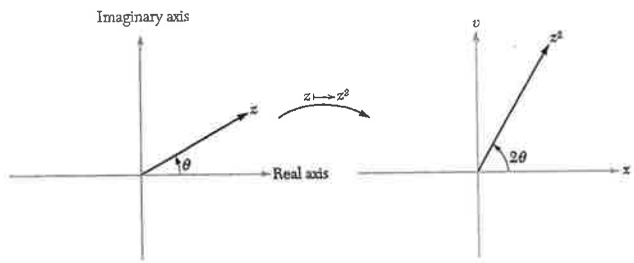

Figure 1.3.3: Squaring function.

From this doubling of angles it follows that the power function z^2 maps the first quadrant to the whole upper half plane (see Figure 1.3.4). Similarly, the upper half plane is mapped to the whole plane.

Now consider the square root function $\sqrt{z} = \sqrt{r}e^{i\theta/2}$. Suppose that we choose a branch by using the interval $0 \le \theta < 2\pi$. Then $0 \le \theta/2 < \pi$, so $\sqrt{z}$ will always lie

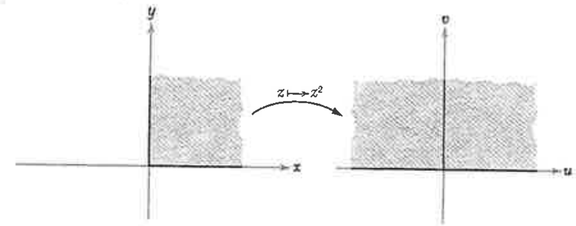

Figure 1.3.4: Effect of the squaring function on the first quadrant.

in the upper half plane, and the angles thus are cut in half. The situation is simila to that involving the exponential function in that $z \mapsto \sqrt{z}$ is the inverse for $z \mapsto z$ when the latter is restricted to a region on which it is one-to-one. In like manne if we choose the branch $-\pi \le \theta < \pi$, we have $-\pi/2 \le \theta/2 < \pi/2$, so $\sqrt{z}$ takes it values in the right half plane instead of the upper half plane. (Generally, any "hal plane" could be used—see Figure 1.3.5.) If we choose a specific branch of $\sqrt{z}$, w also choose which of the two possible square roots we shall obtain.

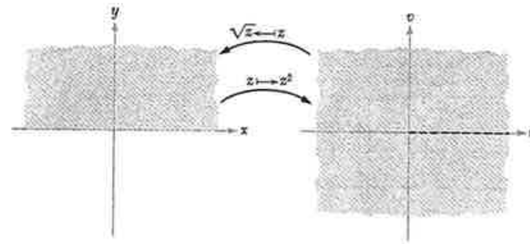

Figure 1.3.5: Squaring function and its inverse.

Various geometric statements can be made concerning the map $z \mapsto z^2$ tha also give information about the inverse, $z \mapsto \sqrt{z}$. For example, a circle of radiu r described by the set of points $re^{i\theta}, 0 \le \theta < 2\pi$, is mapped to $r^2e^{i2\theta}$, a circle o radius r^2; as $re^{i\theta}$ moves once around the first circle, the image point moves twic around (see Figure 1.3.6). The inverse map does the opposite: as z moves alon the circle $re^{i\theta}$ of radius r, $\sqrt{z}$ moves half as fast along the circle $\sqrt{r}e^{i\theta/2}$ of radiu $\sqrt{r}$.

Domains on which $z \mapsto e^z$ and $z \mapsto \log z$ are inverses have already been discusse (see Figure 1.3.2). Note that the lines $y =$ constant, described by the points $x + iy$ as x varies, are mapped by the function $z \mapsto e^z$ to points $e^x e^{iy}$, which is a ray with

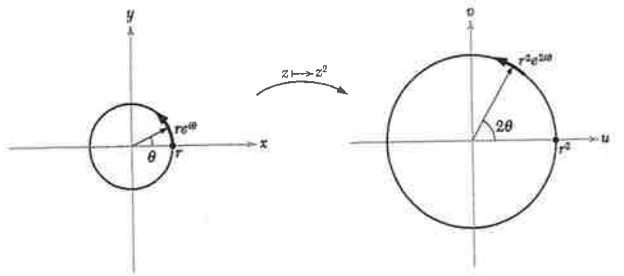

Figure 1.3.6: Effect of the squaring function on a circle of radius r.

argument y. As x ranges from $-\infty$ to $+\infty$, the image point on the ray goes from 0 out to infinity (see Figure 1.3.7).

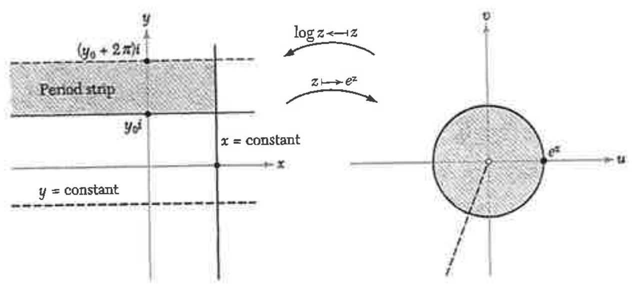

Figure 1.3.7: Geometry of e^z and $\log z$.

Similarly, the vertical line $x =$ constant is mapped to a circle of radius e^x. If we restrict y to an interval of length 2π, the image circle is described once, but if y is unrestricted, the image circle is described infinitely many times as y ranges from $-\infty$ to $+\infty$. The logarithm, being the inverse of e^z, maps points in the opposite direction to e^z, as shown in Figure 1.3.7. Because of the special nature of the striplike regions in Figures 1.3.2 and 1.3.7 (on them e^z is one-to-one) and because of the periodicity of e^z, these regions deserve a name. They are usually called ***period strips*** of e^z.

Worked Examples

Example 1.3.13 *Find the real and imaginary parts of* $\exp(e^z)$. *(It is common to use* $\exp w$ *as another way of writing* e^w*.)*

Solution Let $z = x + iy$; then $e^z = e^x \cos y + ie^x \sin y$. Thus,

$$\exp e^z = e^{e^x \cos y}[\cos(e^x \sin y) + i \sin(e^x \sin y)].$$

Therefore,

$$\operatorname{Re}(\exp e^z) = (e^{e^x \cos y}) \cos(e^x \sin y) \quad \text{and} \quad \operatorname{Im}(\exp e^z) = (e^{e^x \cos y}) \sin(e^x \sin y).$$

Example 1.3.14 *Find all the values of* i^i.

Solution

$$i^i = e^{i \log i} = e^{i[\log 1 + (i\pi/2) + (2\pi n)i]} = (e^{-2\pi n})e^{-\pi/2} = e^{-2\pi(n+1/4)}.$$

All the values of i^i are given by the last expression as n takes integral values, $n = 0, \pm 1, \pm 2, \ldots$.

Example 1.3.15 *Solve* $\cos z = \frac{1}{2}$ *for* z.

Solution We know that $z_n = \pm(\pi/3 + 2\pi n)$, where n is an integer, solves the equation $\cos z = \frac{1}{2}$; we shall show that $z_n, n = 0, \pm 1, \ldots$, are the *only* solutions; that is, there are no solutions off the real axis. We are given

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}.$$

Therefore, $e^{2iz} - e^{iz} + 1 = 0$, and so by the quadratic formula, $e^{iz} = \frac{1}{2} \pm \sqrt{3}i/2$. Hence $iz = \log(\frac{1}{2} \pm \sqrt{3}i/2) = \pm \log(\frac{1}{2} + \sqrt{3}i/2)$, since $\frac{1}{2} + \sqrt{3}i/2$ and $\frac{1}{2} - \sqrt{3}i/2$ are checked to be reciprocals of one another. We thus obtain

$$z = \pm i \log\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \pm i\left(\log 1 + \frac{\pi}{3}i + 2\pi n i\right) = \pm\left(\frac{\pi}{3} + 2\pi n\right).$$

Example 1.3.16 *Consider the mapping* $z \mapsto \sin z$. *Show that lines parallel to the real axis are mapped to ellipses and that lines parallel to the imaginary axis are mapped to hyperbolas.*

Solution Using Proposition 1.3.4 (also see Example 1.3.14), we get

$$\begin{aligned}\sin z &= \sin(x+iy) = \sin x\cos(iy) + \sin(iy)\cos x \\ &= \sin x\cosh y + i\sinh y\cos x\end{aligned}$$

where

$$\cosh y = \frac{e^y + e^{-y}}{2} \quad \text{and} \quad \sinh y = \frac{e^y - e^{-y}}{2}.$$

Suppose that $y = y_0$ is constant; if we write $\sin z = u + iv$, then we have

$$\frac{u^2}{\cosh^2 y_0} + \frac{v^2}{\sinh^2 y_0} = 1$$

since $\sin^2 x + \cos^2 x = 1$. This is an ellipse.

Similarly, if $x = x_0$ is constant, from $\cosh^2 y - \sinh^2 y = 1$ we obtain

$$\frac{u^2}{\sin^2 x_0} - \frac{v^2}{\cos^2 x_0} = 1,$$

which is a hyperbola.

Exercises

1. Express in the form $a + bi$:
   - • (a) e^{2+i}
   - (b) $\sin(1+i)$
2. Express in the form $a + bi$:
   - (a) e^{3-i}
   - (b) $\cos(2+3i)$
3. Solve
   - (a) $\cos z = \frac{3}{4} + \frac{i}{4}$
   - (b) $\cos z = 4$
4. Solve
   - (a) $\sin z = \frac{3}{4} + \frac{i}{4}$
   - • (b) $\sin z = 4$
5. • Find all the values of
   - • (a) $\log 1$
   - (b) $\log i$

6. Find all the values of
   - • (a) $\log(-i)$
   - (b) $\log(1+i)$
7. Find all the values of
   - • (a) $(-i)^i$
   - (b) $(1+i)^{1+i}$
8. Find all the values of
   - (a) $(-1)^i$
   - (b) 2^i
9. • For what values of z is $\overline{(e^{iz})} = e^{i\bar{z}}$?
10. • Let $\sqrt{\cdot}$ denote the particular square root defined by

$$\sqrt{r(\cos\theta + i\sin\theta)} = r^{1/2}[\cos(\theta/2) + i\sin(\theta/2)], 0 \le \theta < 2\pi;$$

the other square root is

$$r^{1/2}\{\cos[(\theta+2\pi)/2] + i\sin[(\theta+2\pi)/2]\}.$$

For what values of z does the equation $\sqrt{z^2} = z$ hold?

11. • Along which rays through the origin (a ray is determined by $\arg z =$ constant) does $\lim_{z\to\infty} |e^z|$ exist?
12. Prove the identity

$$z = \tan\left[\frac{1}{i}\log\left(\frac{1+iz}{1-iz}\right)^{1/2}\right].$$

13. Simplify e^{z^2}, e^{iz}, and $e^{1/z}$, where $z = x + iy$. For $e^{1/z}$ we specify that $z \neq 0$.
14. Examine the behavior of e^{x+iy} as $x \to \pm\infty$ and the behavior of e^{x+iy} as $y \to \pm\infty$.
15. • Prove that $\sin(-z) = -\sin z$; $\cos(-z) = \cos z$; $\sin(\pi/2 - z) = \cos z$.
16. Define sinh and cosh on all of $\mathbb{C}$ by $\sinh z = (e^z - e^{-z})/2$ and $\cosh z = (e^z + e^{-z})/2$. Prove that
   - (a) $\cosh^2 z - \sinh^2 z = 1$
   - • (b) $\sinh(z_1 + z_2) = \sinh z_1\cosh z_2 + \cosh z_1\sinh z_2$
   - (c) $\cosh(z_1 + z_2) = \cosh z_1\cosh z_2 + \sinh z_1\sinh z_2$

(d) $\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$

(e) $\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$

17.• Use the equation $\sin z = \sin x \cosh y + i \sinh y \cos x$ where $z = x+iy$ to prove that $|\sinh y| \le |\sin z| \le |\cosh y|$.

18. If b is real, prove that $|a^b| = |a|^b$.

19. Is it true that $|a^b| = |a|^{|b|}$ for all $a, b \in \mathbb{C}$?

20. (a) For complex numbers a, b, c, prove that $a^b a^c = a^{b+c}$, using a fixed branch of log.

(b) Show that $(ab)^c = a^c b^c$ if we choose branches so that $\log(ab) = \log a + \log b$ (with no extra $2\pi n i$).

21.• Using polar coordinates, show that $z \mapsto z + 1/z$ maps the circle $|z| = 1$ to the interval $[-2, 2]$ on the x axis.

22. (a) The map $z \mapsto z^3$ maps the first quadrant onto what?

(b) Discuss the geometry of $z \mapsto \sqrt[3]{z}$ as was done in the text for $\sqrt{z}$.

23.• The map $z \mapsto 1/z$ takes the exterior of the unit circle to the interior (excluding zero) and vice versa. To what are lines $\arg z =$ constant mapped?

24. What are the images of vertical and horizontal lines under $z \mapsto \cos z$?

25. Under what conditions does $\log a^b = b \log a$ for complex numbers a, b? (Use the branch of log with $-\pi \le \theta < \pi$.)

26. (a) Show that under the map $z \mapsto z^2$, lines parallel to the real axis are mapped to parabolas.

(b) Show that under (a branch of) $z \mapsto \sqrt{z}$, lines parallel to the real axis are mapped to hyperbolas.

27. Show that the n nth roots of unity are $1, w, w^2, w^3, \ldots, w^{n-1}$, where $w = e^{2\pi i/n}$.

28. Show that the trigonometric identities can be deduced if $e^{i(x_1+x_2)} = e^{ix_1} \cdot e^{ix_2}$ is assumed.

29.• Show that $\sin z = 0$ iff $z = k\pi, k = 0, \pm 1, \pm 2, \ldots$.

30. Show that the sine and cosine are periodic with minimum period 2π; that is, that

(a) $\sin(z + 2\pi) = \sin z$ for all z.

•(b) $\cos(z + 2\pi) = \cos z$ for all z.

(c) $\sin(z + \omega) = \sin z$ for all z implies $\omega = 2\pi n$ for some integer n.

(d) $\cos(z + \omega) = \cos z$ for all z implies $\omega = 2\pi n$ for some integer n.

31. Find the maximum of $|\cos z|$ on the square

$$0 \le \operatorname{Re} z \le 2\pi, 0 \le \operatorname{Im} z \le 2\pi.$$

32. Show that $\log z = 0$ iff $z = 1$, using the branch with $-\pi < \arg z \le \pi$.

33. Compute the following quantities numerically to two significant figures:
(a) $e^{3.2+6.1i}$ (b) $\log(1.2 - 3.0i)$ (c) $\sin(8.1i - 3.2)$

34.• Show that the function $\sin z$ maps the strip $-\pi/2 < \operatorname{Re} z < \pi/2$ onto the s $\mathbb{C}\backslash\{z \mid \operatorname{Im} z = 0 \text{ and } |\operatorname{Re} z| \ge 1\}$.

35.• Discuss the inverse functions $\sin^{-1} z$ and $\cos^{-1} z$. For example, is $\sin z$ or to-one on the set defined by $0 \le \operatorname{Re} z < 2\pi$?

1.4 Continuous Functions

In this section and the next, the fundamental notions of continuity and differe tiability for complex-valued functions of a complex variable will be analyzed. T results are similar to those learned in the calculus of functions of real variabl These sections will be concerned mostly with the underlying theory, which is a plied to the elementary functions in §1.6.

Since $\mathbb{C}$ is $\mathbb{R}^2$ with the extra structure of complex multiplication, many geomet concepts can be translated from $\mathbb{R}^2$ into complex notation. This has already be done for the absolute value, $|z|$, which is the same as the norm, or length, of regarded as a vector in $\mathbb{R}^2$. Furthermore, we will use calculus for functions of t variables in the study of functions of a complex variable.

Open Sets We will need the notion of an open set. A set $A \subset \mathbb{C} = \mathbb{R}^2$ is call ***open*** when, for each point z_0 in A, there is a real number $\epsilon > 0$ such that $z \in$ whenever $|z - z_0| < \epsilon$. See Figure 1.4.1. The value of ϵ may depend on z_0; as gets close to the "edge" of A, ϵ gets smaller. Intuitively, a set is open if it does n contain any of its "boundary" or "edge" points.

For a number $r > 0$, the r ***neighborhood*** or r ***disk*** around a point z_0 in $\mathbb{C}$ defined to be the set $D(z_0; r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$. For practice, the stude should prove that for each $w_0 \in \mathbb{C}$ and $r > 0$, the disk $A = \{z \in \mathbb{C} \mid |z - w_0| <$ is itself open. A ***deleted r neighborhood*** is an r neighborhood whose center poi has been removed. Thus a deleted r-neighborhood has the form $D(z_0; r)\backslash\{z$ which stands for the set $D(z_0; r)$ minus the singleton set $\{z_0\}$. See Figure 1.4.2.

A ***neighborhood*** of a point z_0 is, by definition, a set containing some r di around z_0. Notice that a set A *is open iff for each z_0 in A, there is an r neighborho of z_0 wholly contained in A.*

The basic properties of open sets are collected in the next proposition.