Name...
I.D. number...

Math 2270-1
Sample Final Exam
SOLUTIONS
December 8, 2000

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 200 points possible, and the point values for each problem are indicated in the right-hand margin. Of course, this exam counts for 30% of your final grade even though it is scaled to 200 points. Good Luck!

```
> restart:with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
>
```

1) Let

$$A := \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Then L(**x**)=A**x** is a matrix map from **R**^3 to **R**^2.

1a) Find the four fundamental subspaces associated to the matrix A.

(20 points)

```
> A:=matrix(2,3,[1,-1,2,1,0,1]);
```

$$A := \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

```
> rref(A);
```

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

So, the row space is the span of {[1,,0,1],0,1,-1]}. The column space is the span of the first two columns of A, and this is all of R^2. The nullspace can be obtained by backsolving the homogeneous system, using rref(A). We see that x3=t, x2=t, x1=-t, so x=t[-1,1,1], so a nullspace basis is {[-1,1,1]} . Since the column space is all or R^2, its orthogonal complement is the zero vector.

1b) State and verify the theorem which relates rank and nullity of A, in this particular case

(5 points)

rank + nullity = #cols(A). In this case 2+1=3.

1c) Find an orthonormal basis for R^3 in which the first two vectors are a basis for the rowspace of A and the last vector spans its nullspace.

(10 points)

The rowspace and nullspace of A are already orthogonal, so I just need to Gramschmidt my rowspace basis, and then normalize:

```
> w1:=vector([1,0,1]);
  w2:=vector([0,1,-1]);
```

$$w1 := [1, 0, 1]$$

$$w2 := [0, 1, -1]$$

```
> v1:=evalm(w1);
  v2:=evalm(w2-(dotprod(w2,v1)/dotprod(v1,v1))*v1);
```

$$v1 := [1, 0, 1]$$

$$v2 := \left[\frac{1}{2}, 1, \frac{-1}{2}\right]$$

So my basis is

$$\left[\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix}\right]$$

1d) Find an implicit equation ax+by+cz = 0 satisfied for precisely those points which are in the rowspace of A.

(5 points)

I can take normal vector as my nullspace basis, so the equation is -x+y+z=0.

2a) Exhibit the rotation matrix which rotates vectors in R^2 by an angle of α radians in the counter-clockwise direction.

(5 points)

$$rot := \alpha \rightarrow \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

\>

2b) Verify that the product of an α-rotation matrix with a β-rotation matrix is an (α+β)-rotation matrix.

(10 points)

$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} =$$

$$\begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \end{bmatrix}$$

By trig addition angle formulas we recognize this last matrix as

$$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

>

3) Find the least-squares line fit through the four points (1,1), (2,1), (-1,0), (-2,0) in the plane.

(10 points)

$$A := \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$b := \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Our least squares solution will solve

$$\begin{bmatrix} 1 & 2 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \frac{3}{10} \\ \frac{1}{2} \end{bmatrix}$$

So the least squares line fit is

$$y = \frac{3}{10} x + \frac{1}{2}$$

4a) Explain the procedure which allows one to convert a general quadratic equation in n-variables

$$x^T A x + B x + c = 0$$

into one without any ‘‘cross terms’’. Be precise in explaining the change of variables, and the justification for why such a change of variables exists.

(10 points)

A is symmetric, so we can diagonalize it with an orthogonal matrix P, ie transpose(P)AP=Lambda. We then let

$$x = Py$$

and the quadratic form transforms to

$$y^T \Lambda y + B P y + c = 0$$

4b) Apply the procedure from part (4a) to put the conic section

$$6 x^2 + 9 y^2 - 4 x y + 4 \sqrt{5}\, x - 18 \sqrt{5}\, y = 5$$

into standard form. Along the way, identify the conic section.

(20 points)

$$A := \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$$

```
> eigenvects(A);
```

$$[10, 1, \{[1, -2]\}], [5, 1, \{[2, 1]\}]$$

Since both eigenvalues are positive, we have an ellipse. Continuing, we can take

$$P := \frac{1}{5}\sqrt{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

And, using u,v for our new variables, our transformed equation is

$$10 u^2 + 5 v^2 + 40 u - 10 v = 5$$

we may complete the square to get

$$5 (v-1)^2 - 45 + 10 (u+2)^2 = 5$$

$$5 (v-1)^2 + 10 (u+2)^2 = 50$$

$$\frac{1}{10} (v-1)^2 + \frac{1}{5} (u+2)^2 = 1$$

5) Let

$$C := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

5a) Find the inverse of C using elementary row operations.

(15 points)

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

```
> rref(%);
```

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & 4 & 3 & -2 \end{bmatrix}$$

So the inverse matrix is

$$\begin{bmatrix} -1 & -1 & 1 \\ -2 & -1 & 1 \\ 4 & 3 & -2 \end{bmatrix}$$

5b) Find the inverse of C using the adjoint formula.

(15 points)

```
> cof(C):=matrix(3,3,[1,2,-4,1,1,-3,-1,-1,2]);
```

$$\operatorname{cof}(C) := \begin{bmatrix} 1 & 2 & -4 \\ 1 & 1 & -3 \\ -1 & -1 & 2 \end{bmatrix}$$

```
> adj(C):=transpose(cof(C));
```

$$\operatorname{adj}(C) := \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -1 \\ -4 & -3 & 2 \end{bmatrix}$$

We calculate that det(C)=2-2-1=-1, so the inverse is given by

```
> detC:=-1;
  (1/detC)*adj(C);
```

$$detC := -1$$

$$-\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -1 \\ -4 & -3 & 2 \end{bmatrix}$$

6a) Define what it means for a function L:V-->W between vector spaces to be a **linear transformation**.

(5 points)

I) L(u+v)=L(u)+L(v) for all u,v in V

II) L(cu)=cL(u) for all u in V, c in R.

6b) Define what it means for a set S={$\mathbf{v}_1$,$\mathbf{v}_2$, . . . $\mathbf{v}_n$} to be a **basis** for a vector space V.

(5 points)

S is a basis for V if it is linearly independent and spans V.

6c) For a linear map L as in part (6a), define the **nullspace (kernel)** of L.

(5 points)

The kernel of L is the set of all vectors in V which satisfy L(v)=0.

6d) Let S={$\mathbf{v}_1$,$\mathbf{v}_2$, . . . $\mathbf{v}_n$} be a basis for V, and T={$\mathbf{w}_1$,$\mathbf{w}_2$, . . . $\mathbf{w}_m$} be a basis for W. Let L:V-->W be linear. Explain what the matrix for L with respect to S and T is, and how to compute it.

(10 points)

*The matrix for L with respect to S and T is the matrix A which has the property that A times the S-coordinates of a vector v in V equals the T-coordinates of L(v), for all v in V. The jth column of A is just the coordinates of L(**v**j), with respect to T. (So that's one way to compute it.)*

7) Let

$$A := \begin{bmatrix} -7 & -6 & 5 \\ 4 & 4 & -2 \\ -6 & -5 & 6 \end{bmatrix}$$

7a) Let

$$T := \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(Note these are the columns of your matrix C from #5.) Let L(x)=Ax, a matrix map from R^3 to R^3. Thus A is the matrix of L with respect to the standard basis of R^3. Find the matrix of L with respect to the T basis.

(20 points)

This means use the T-basis in both the domain and range space.

Method I: The matrix for L with respect to T is the triple product $[\mathbf{P}_{T\text{-}<E}][\mathbf{A}][\mathbf{P}_{E<\text{-}T}]$

```
> A:=matrix(3,3,[-7,-6,5,4,4,-2,-6,-5,6]);
```

$$A := \begin{bmatrix} -7 & -6 & 5 \\ 4 & 4 & -2 \\ -6 & -5 & 6 \end{bmatrix}$$

```
> PET:=matrix(3,3,[1,-1,0,0,2,1,2,1,1]);
```

$$PET := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

```
> evalm(inverse(PET))*evalm(A)*evalm(PET);
>
```

$$\begin{bmatrix} -1 & -1 & 1 \\ -2 & -1 & 1 \\ 4 & 3 & -2 \end{bmatrix} \begin{bmatrix} -7 & -6 & 5 \\ 4 & 4 & -2 \\ -6 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

```
> evalm(inverse(PET)&*A&*PET);
```

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

Method 2: For T={v1,v2,v3} compute {Av1,Av2,Av3} and then find coords with respect to T. It turns out that Av1=3v1, Av2=2v3, Av3=v2. So it is easy to write down the matrix for L exhibited above.

In general, this leads to the augmented matrix

```
> matrix(3,6,[1,-1,0,3,0,-1,0,2,1,0,2,2,2,1,1,6,2,1]);
```

$$\begin{bmatrix} 1 & -1 & 0 & 3 & 0 & -1 \\ 0 & 2 & 1 & 0 & 2 & 2 \\ 2 & 1 & 1 & 6 & 2 & 1 \end{bmatrix}$$

```
> rref(%);
```

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

From which we take the last 3 columns as our matrix.

7b) Let

$$v := 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Compute L(v) two ways: once using the matrix for L with respect to T, and once using the matrix A. Verify that your answers agree.

(10 points)

Method 1: Use the E-coordinates for v

```
> evalm(A)*matrix(3,1,[1,1,4])=evalm(A&*matrix([[1], [1], [4]]));
```

$$\begin{bmatrix} -7 & -6 & 5 \\ 4 & 4 & -2 \\ -6 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 13 \end{bmatrix}$$

Method 2: Use T-coordinates

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 13 \end{bmatrix}$$

8) True-False. Two points each. No justification required!

(20 points)

8a) Let A be an n by n matrix. Then if Ax=Ay it follows that x=y.
FALSE: would only be true if A was non-singular
8b) If A and B are n by n matrices, then

$$(A+B)^2 = A^2 + 2\,A\,B + B^2$$

FALSE: would only be true if AB=BA
8c) If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are orthogonal to $\mathbf{w}$, then any vector in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is also orthogonal to $\mathbf{w}$.
TRUE: if the dot product of each vi with w is zero, then the dot product of any linear combination of the vj's with w is also zero.
8d) Every set of five orthonormal vectors in **R**^5 is automatically a basis for **R**^5.
TRUE: orthonormal vectors are automatically linearly independent, and 5 independent vectors automatically span a 5-dimensional space
8e) The number of linearly independent eigenvectors of a matrix is always greater than or equal to the number of distinct eigenvalues.
TRUE: each eigenspace is at least one-dimensional, and the union of eigenspace bases is still linearly independent.
8f) If the rows of a 4 by 6 matrix are linearly dependent then the nullspace is at least three dimensional.
TRUE: rows dependent so row rank is at most 3. Since row rank plus nullity equals 6 this means nullity is at least 3
8g) A diagonalizable n by n matrix must always have n distinct eigenvalues.
FALSE: e.g. the identity matrix has only one distinct eigenvalue (=1), but is diagonal
8h) Every orthogonal matrix is diagonalizable.
FALSE: e.g. rotation matrices have no real eigenvectors (unless the rotation is a multiple of Pi).
8i) If A and B are orthogonal matrices then so is AB.
TRUE: use the fact that the transpose of AB is B transpose time A transpose.
8j) If A is a square matrix and A^2 is singular, then so is A.
TRUE: if det(A^2)=0, then this also equals (det(A))^2 by multiplicative determinant property, so det(A)=0.