Name________________________

Student I.D.__________________

Math 2250-1
Quiz 10
November 16, 2012

Directions: Because the homework was split evenly between Laplace transforms and eigenvectors this week, you may choose either problem 1 or problem 2 below to complete. If you attempt both, make it very clear which one you want graded.

1) Consider the matrix

$$A := \begin{bmatrix} 4 & 9 \\ -4 & -8 \end{bmatrix}.$$

a) Find the eigenvalues and eigenvectors (eigenspace bases).

(8 points)

b) Is this matrix diagonalizable? Explain why or why not.

(2 points)

a) *The characteristic polynomial is*

$$\begin{vmatrix} 4-\lambda & 9 \\ -4 & -8-\lambda \end{vmatrix} = (4-\lambda)(-8-\lambda) + 36 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 .$$

Thus $\lambda = -2$ is a double root, and the only eigenvalue. We find an eigenbasis for $E_{\lambda=-2}$ by reducing the system

$$\left[\begin{array}{cc|c} 6 & 9 & 0 \\ -4 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Method 1: *We can backsolve for* $[v_1, v_2]$, yielding $v_2 = t$, $v_1 = -\frac{3}{2}t$ so

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

So we may take a basis for this one-dimensional eigenspace by letting $t = 2$ (to clear fractions), i.e. the eigenvector

$$\underline{\mathbf{v}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

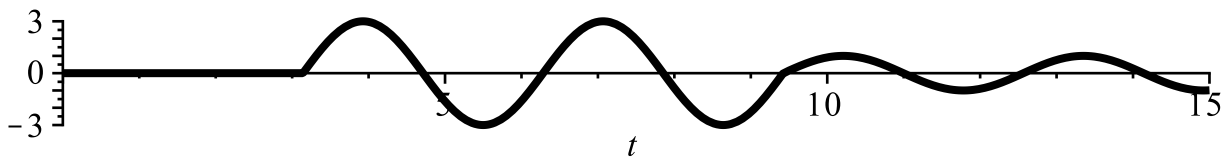
Method 2: *Since homogeneous solutions to matrix equations correspond to column dependencies, we see from the first reduction that $-3\, col_1 + 2\, col_2$ holds for the original (and reduced) matrices, so $\underline{\mathbf{v}} = [-3, 2]^T$ is an eigenvector. Since the solution space is one-dimensional (one column without a leading 1, so one free parameter), this eigenvector is a basis for $E_{\lambda=-2}$.*

b) *No, this matrix is not diagonalizable. There are not two linearly independent eigenvectors, i.e. there is not a basis for $\mathbb{R}^2$ consisting of eigenvectors of A.*

2) Solve the initial value problem below for an undamped mass-spring configuration subject to impulse forces at time $t = \pi$ and $t = 3\pi$. (There is a Laplace transform table on the back of this quiz.)

$$x''(t) + 4\,x(t) = 3\cdot\delta(t-\pi) - 2\cdot\delta(t-3\pi)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Hint: The solution has this graph:



(10 points)

<u>*Solution:*</u> *Taking Laplace transform for the IVP yields*

$$s^2X(s) + 4\,X(s) = 3\,e^{-\pi s} - 2\,e^{-3\pi s}$$
$$X(s)\left(s^2+4\right) = 3\,e^{-\pi s} - 2\,e^{-3\pi s}\,.$$
$$X(s) = \frac{3\,e^{-\pi s}}{s^2+4} - \frac{2\,e^{-3\pi s}}{s^2+4}\,.$$

So using the t-translation entry

$$x(t) = 3\,u(t-\pi)\left(\frac{1}{2}\sin(2(t-\pi))\right) - u(t-3\pi)\sin(2(t-3\pi))\,.$$

Simplifying, and noticing that the translations in t are by multiples of the period of $\sin(2\,t)$ we get

$$x(t) = u(t-\pi)\left(\frac{3}{2}\sin(2\,t)\right) - u(t-3\pi)\sin(2\,t)\,.$$

We can also express this function using the "piecewise" notation:

$$x(t) = \begin{cases} 0, & 0 \le t < \pi \\ \frac{3}{2}\sin(2\,t), & \pi \le t < 3\pi \\ \frac{1}{2}\sin(2\,t). & t \ge 3\pi \end{cases}\,.$$

<u>***Note:***</u> *As one of you astutely pointed out after the quiz was handed in, the vertical axis in the suggestive "hint" graph was incorrectly scaled by a factor of 2.*