Math 2250-1
Friday 9/9

§2.3 improved velocity/acceleration models.

for motion along a line, with position $x(t)$ (or $y(t)$)

$x'(t) = v(t)$ velocity
$v'(t) = a(t)$ acceleration

Newton's 2nd Law:

$$m\frac{dv}{dt} = F \quad \text{where } F \text{ is the net force}$$

Cases we've seen:

- $m\frac{dv}{dt} = m\alpha$ with α constant

$$\frac{dv}{dt} = \alpha$$
$$v = \alpha t + v_0$$
$$x = \tfrac{1}{2}\alpha t^2 + v_0 t + x_0$$

applications: objects thrown vertically in a constant gravitational field ($\alpha = -g$, for instance)

objects (cars e.g.) subject to constant acceleration or deceleration.

drawback: doesn't account for frictional forces

Add resistance:

$$m\frac{dv}{dt} = m\alpha + \underset{\uparrow}{F_f}$$

$|F_f| \cong k|v|^p$ $\quad 1 \le p \le 2$ is an empirical law that is sometimes fitted to data

$p = 1$: linear model

$$m\frac{dv}{dt} = m\alpha \underset{\uparrow}{-} kv$$

because friction acts to decelerate.

if $F_f = F_f(v) = \cancel{F(0)}^{\,0} + F'(0)v + \frac{1}{2!}F''(0)v^2 + \dots$

this is a "small v" linearization model.

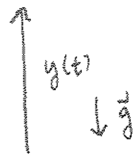
$p = 2$: $\quad m\frac{dv}{dt} = m\alpha - kv^2$ if $v > 0$

$m\frac{dv}{dt} = m\alpha + kv^2$ if $v < 0$!

focus on these models for vertical motion on earth.

If we choose up to be the positive direction then

p=1 : $m \frac{dv}{dt} = -mg - kv$



(how would DE change if down is chosen as the positive y direction?)

$$\frac{dv}{dt} = -g - \rho v \qquad \rho = \frac{k}{m}$$

$$\frac{dv}{dt} = -\rho\left(v + \frac{g}{\rho}\right)$$

equilibrium soln for velocity is $v = -\frac{g}{\rho}$.

- Draw the phase diagram and slope field for this DE

- What do you predict for solns to the

$$\text{IVP} \begin{cases} \frac{dv}{dt} = -\rho\left(v + g/\rho\right) \\ v(0) = v_0 \end{cases}$$

as $t \to \infty$.

analytic sol'n, find it!

$$\text{IVP}\begin{cases}\dfrac{dv}{dt} = -\rho\left(v + \dfrac{g}{\rho}\right)\\ v(0) = v_0\end{cases}$$

sol'n:

$$\boxed{v = \underbrace{-g/\rho}_{v_T} + \left(v_0 + g/\rho\right)e^{-\rho t}}$$

v_T "terminal velocity"

$$y = \int v(t)\,dt = t v_T + \left(\frac{v_0 - v_T}{-\rho}\right)e^{-\rho t} + C$$

$$y_0 = \frac{v_T - v_0}{\rho} + C$$

so $C = y_0 + \dfrac{v_0 - v_T}{\rho}$

$$\boxed{y = t v_T + \left(\frac{v_0 - v_T}{\rho}\right)\left(1 - e^{-\rho t}\right) + y_0}$$

Examples 1 & 2 p. ~~98-100~~ 100-102

crossbow bolt

$v_0 = 49$ m/s

$g = 9.8$ m/s²

$y_0 = 0$

no friction

$y = -4.9t^2 + 49t$

$v = -9.8t + 49$

max ht at $t = 5$ sec

$y_{max} = y(5) = 49(2.5)$

$= 122.5$ m

time aloft = 10 sec.

linear drag

$\rho = .04$ (drag coeff; empirical)

corresponds to

$|v_T| = \dfrac{g}{\rho} = \dfrac{9.8}{.04} = 245$ m/sec

(you could measure this to deduce ρ)

$v_0 - v_T = 49 + 245 = 294$

so

$v = -245 + 294e^{-.04t}$

$v = 0$ at $\dfrac{245}{294} = e^{-.04t}$

$t_{max} \approx 4.56$ sec

$y = -245t + (294)(25)\left(1 - e^{-.04t}\right)$

$y(t_{max}) = ?$

When does bolt hit ground?

Computation sheet for examples 1–2 section 2.3

```
> restart :
  Digits := 5 :
```

Example 1:

```
> a1 := -9.8;
  v1 := t -> -9.8·t + 49;
  y1 := t -> -4.9·t^2 + 49·t;
```

$$a1 := -9.8$$
$$v1 := t \rightarrow (-1) \cdot 9.8\, t + 49$$
$$y1 := t \rightarrow (-1) \cdot 4.9\, t^2 + 49\, t \tag{1}$$

```
>
```

Example 2: First, check our hand work:

```
> deqtn := diff(v(t), t) = -rho·(v(t) + g/rho);
  ics := v(0) = v0; # if I'd used a subscript "0" there would be issues for v(t) later.
```

$$deqtn := \frac{d}{dt}\, v(t) = -\rho \left(v(t) + \frac{g}{\rho} \right)$$
$$ics := v(0) = v0 \tag{2}$$

```
> with(DEtools) :
  dsolve({deqtn, ics});
```

$$v(t) = -\frac{g}{\rho} + e^{-\rho t} \left(v0 + \frac{g}{\rho} \right) \tag{3}$$

```
> v := t -> vτ + e^(-ρt) (v0 - vτ);  # vτ= -g/ρ
  deqtn2 := diff(y(t), t) = v(t);
  ics2 := y(0) = y0;
  dsolve({deqtn2, ics2});
```

$$v := t \rightarrow v\tau + e^{-\rho t} (v0 - v\tau)$$
$$deqtn2 := \frac{d}{dt}\, y(t) = v\tau + e^{-\rho t} (v0 - v\tau)$$
$$ics2 := y(0) = y0$$
$$y(t) = -\frac{e^{-\rho t} (v0 - v\tau)}{\rho} + v\tau\, t + y0 + \frac{v0 - v\tau}{\rho} \tag{4}$$

```
> y := t -> -e^(-ρt) (v0 - vτ)/ρ + vτ t + y0 + (v0 - vτ)/ρ;
```

$$y := t \rightarrow -\frac{e^{-\rho t} (v0 - v\tau)}{\rho} + v\tau\, t + y0 + \frac{v0 - v\tau}{\rho} \tag{5}$$

Now plug in our values:

```
> v0 := 49.:
  g := 9.8:
  y0 := 0:
  ρ := .04:
  vτ := - g/ρ:
  v(t);
  y(t);
```

$$v0 := 49.$$
$$g := 9.8$$
$$y0 := 0$$
$$\rho := 0.04$$
$$v\tau := -245.00$$
$$-245.00 + 294.00\, e^{-0.04\, t}$$
$$-7350.0\, e^{-0.04\, t} - 245.00\, t + 7350.0 \quad (6)$$

```
> solve(v(t) = 0.0, t);
```

$$4.5580 \quad (7)$$

```
> y(4.5580); # maximum height
```

$$108.3 \quad (8)$$

```
> solve(y(t) = 0.0, t); # should give landing time
```

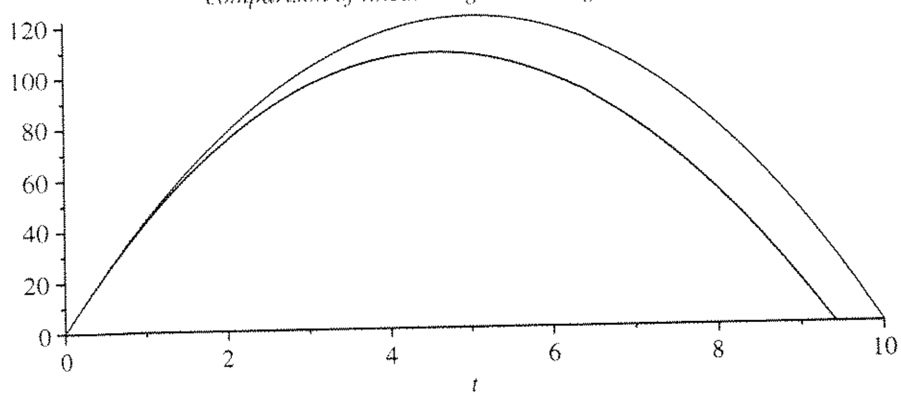
$$9.4110,\ 0. \quad (9)$$

```
> 9.4110 - 4.5580; # falling time
```

$$4.8530 \quad (10)$$

```
> with(plots) :
  plot1 := plot(y1(t), t = 0..10, color = green) :
  plot2 := plot(y(t), t = 0..9.4110, color = blue) :
  display( {plot1, plot2}, title = 'comparison of linear drag vs no drag models');
```

comparison of linear drag vs no drag models



quadratic drag is also interesting...

going up:

$$m \frac{dv}{dt} = -mg - kv^2$$

$$\frac{dv}{dt} = -g\left(1 + \frac{\rho}{g} v^2\right)$$

$$\frac{dv}{1 + \frac{\rho}{g} v^2} = -g\, dt \ ..$$

arctan!

going down:

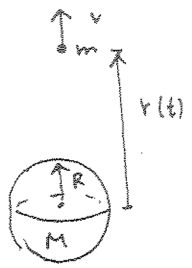
$$m \frac{dv}{dt} = -mg + kv^2$$

$$\frac{dv}{dt} = -g\left(1 - \frac{\rho}{g} v^2\right)$$

$$\frac{dv}{1 - \frac{\rho}{g} v^2} = -g\, dt$$

partial fractions!

A different improvement: A projectile with a really big v_0, so $F=-mg$ no longer operative



$$F = \frac{-mMG}{r^2} \quad \text{instead.}$$

$$\cancel{m}\frac{dv}{dt} = F = -\frac{\cancel{m}MG}{r^2}$$

trick: $\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$

$$v\frac{dv}{dr} = \frac{-GM}{r^2}$$

$$v\,dv = \frac{-GM}{r^2}\,dr$$

$$\frac{1}{2}v^2 = \frac{GM}{r} + C$$

$$v(R) = v_0 \Rightarrow \frac{1}{2}v_0^2 = \frac{GM}{R} + C \Rightarrow C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$\Rightarrow \frac{1}{2}v^2 = GM\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}v_0^2$$

$$v^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right) + v_0^2$$

I forgot this 2 in the original notes ⍨

if there is a maximum ht, it occurs when $v=0$, so at r s.t.

$$2GM\left(\frac{1}{r} - \frac{1}{R}\right) + v_0^2 = 0$$

in order for $r \to \infty$ must have

$$2GM\left(\frac{1}{r} - \frac{1}{R}\right) + v_0^2 > 0 \quad \forall r$$

i.e.

$$2GM\left(-\frac{1}{R}\right) + v_0^2 \geq 0 \qquad (\text{let } r \to \infty)$$

$$v_0^2 = \frac{2GM}{R}$$

$v_0 = \sqrt{\frac{GM2}{R}}$ is called the <u>escape velocity</u>