Math 2250-1
Wed 9/7

§2.2

Recall

- autonomous 1^{st} order DE

- equilibrium sol'ns

- stability
  unstable
  asymptotically stable

additional examples: find equilibria; discuss stability

- $\frac{dx}{dt} = x^4 - x^2$

- $\frac{dx}{dt} = \cos x$

- did anyone think of an example of a 1^{st} order autonomous DE with an equilibrium sol'n which is <u>stable</u>, but <u>not</u> asymptotically stable?

Fri $\otimes$ page (2) today.

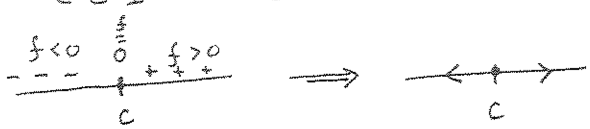
<u>Theorem</u> Consider the autonomous differential equation

$$\frac{dx}{dt} = f(x)$$

with f and $\frac{\partial f}{\partial x}$ continuous. (so local existence and uniqueness theorem holds).

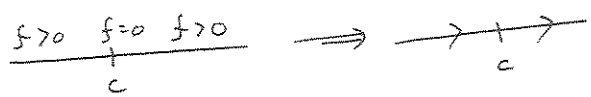
Let $f(c) = 0$ so $x(t) \equiv c$ is an equilibrium sol'n.

Suppose c is an isolated zero of f, i.e c is in an open interval I, $c \in I$ and f has no other zeroes in I. Then



unstable

asymptotically stable

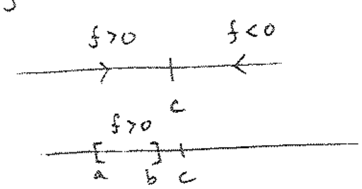


unstable (half stable)

unstable (half stable).

you can prove this theorem with Calculus !! (want to try?)

e.g. consider the second case



f cont; $f > 0$ on subinterval $[a,b]$

$\Rightarrow f \geq \delta > 0$ on $[a,b]$

(extreme value thm from calculus, f attains its minimum)

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a,b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$ ■

Applications

doomsday extinction:

$$\frac{dP}{dt} = aP^2 - bP \qquad a, b > 0.$$
$$= kP(P-M) \qquad \begin{matrix} k = a \\ -kM = b \end{matrix}$$

e.g. if $\beta(t)$ (fertility rate) $\overset{\text{proportional to}}{\sim} P$

(e.g. herd roams in a fixed area and mates randomly upon meeting opposite-sex mate)

then $B(t) = aP^2$;

perhaps $D(t) = -bP$.

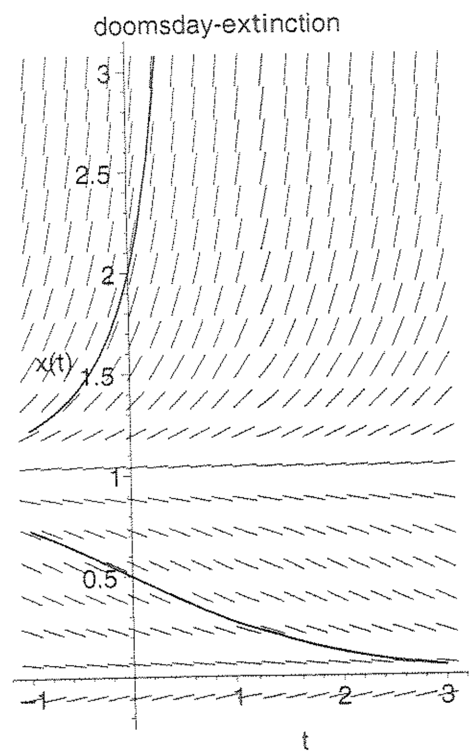
$+ + + \; 0 \; - - - - - \; 0 \; + +$ sign$(f(P))$

$0 \qquad M$

0 stable; M Unstable!

If $0 < P_0 < M$ then $\lim_{t\to\infty} P(t) = 0$ (so in real life, extinction)

If $P_0 > M$ then $\exists \; t_1 < \infty$ s.t. $\lim_{t \to t_1} P(t) = +\infty$ (doomsday!)



example

$$\begin{cases} \dfrac{dx}{dt} = x(x-1) \\ x(0) = 2 \end{cases}$$

Find the time of doomsday.

(ans $t = \ln 2$!)

harvesting a logistic population (e.g. a fishery).

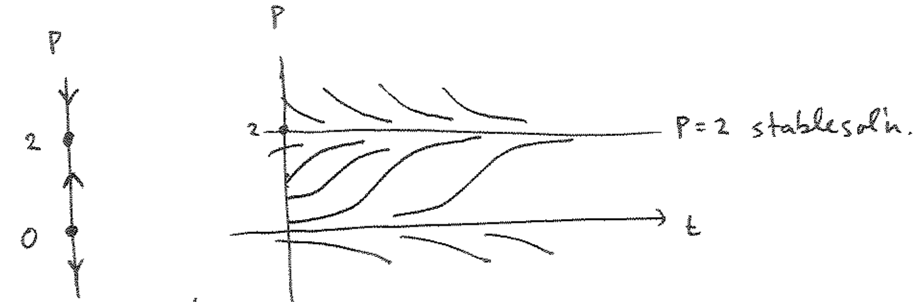
$$\frac{dP}{dt} = \underbrace{aP - bP^2}_{\text{logistic}} \; \underbrace{-\,h}_{}$$

constant rate harvesting. See examples 4-6 §2.2
one could also analyze "constant effort harvesting"
by considering instead a term $-hP$, see e.g. §2.2 #23.

for computational simplicity
take $a=2$, $b=1$

Case 0 no harvesting

$$P'(t) = 2P - P^2$$
$$= P(2-P)$$



phase portrait.

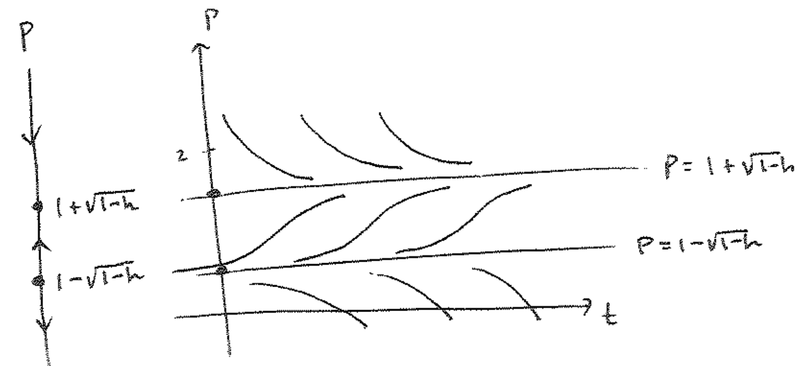
if $P_0 > 0$, $P(t) \to 2$.
(units could be millions of fish, e.g.).

with harvesting:

$$P'(t) = 2P - P^2 - h$$
$$= -(P^2 - 2P + h)$$
$$= -(P-P_1)(P-P_2)$$
$$P_1, P_2 = \frac{2 \pm \sqrt{4-4h}}{2}$$
$$= 1 \pm \sqrt{1-h}$$

Case 1: subcritical harvesting

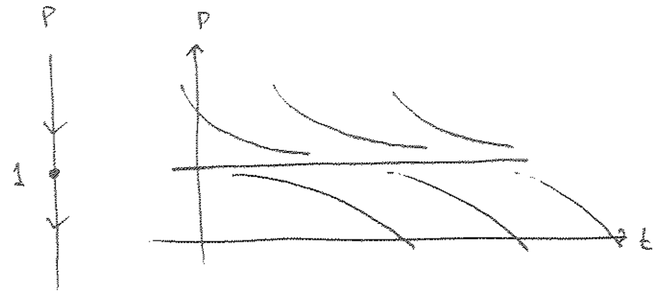
$$0 < h < 1$$



Case 2 Critical harvesting

$h = 1$

$$P'(t) = -(P-1)^2$$

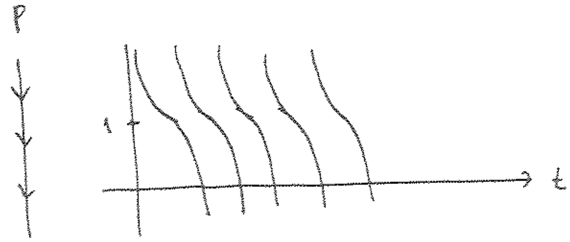


Case 3 Over harvesting

$h > 1$

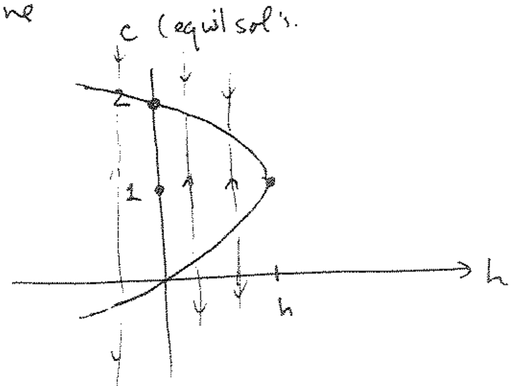
complex roots.

$$P'(t) = -(P^2 - 2P + h)$$
$$= -[(P-1)^2 + (h-1)]$$
$$< 0.$$



this model gives a plausible explanation for why many fisheries collapse: if $h < 1$ but near 1, and if something perturbs the system a little bit (e.g. slightly increased fishing pressure, a big storm, etc.), you could be confronted with "sudden" unexpected fishery collapse.

"bifurcation diagram" of equilibrium sol'ns in the h–c plane



(you can fill in the vertical phase portraits as I indicated. negative h might model constant rate fish stocking...)