Math 2250-1
Tues 9/20
§ 3.1 - 3.3

linear system of m eqtns in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + \quad & a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + \quad & a_{2n}x_n = b_2 \\ a_{31}x_1 + \cdots \quad & \\ \vdots \quad & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots \quad & + a_{mn}x_n = b_m \end{aligned}$$

rewritten in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e. $A\vec{x} = \vec{b}$

A = coefficient matrix
$\vec{b}$ = RHS vector.

in other words,

$$\text{entry}_i(A\vec{x}) := a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$
$$= \text{row}_i(A) \cdot \vec{x} \quad \text{(dot product)}.$$

[only works if the vector $\vec{x}$ has as many entries as A has columns]

<u>exercise</u>: $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} =$ $\qquad$ $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$

Check and record ~~two~~ important algebra properties

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$$

$$A(k\vec{x}) = kA\vec{x}$$
k a scalar

Summarize our discussion yesterday about criteria for solvability of

$$A\vec{x} = \vec{b}$$

in terms of r.r.e.f. (A)

Here's a "typical" rref(A) to motivate discussion

$$\begin{bmatrix} 1 & * & 0 & 0 & * & 0 \\ 0 & 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $A\vec{x} = \vec{b}$ always has sol'tns $\vec{x}$ if and only if ("iff")

  i.e. no matter what $\vec{b}$ is

- Solutions $\vec{x}$ to $A\vec{x} = \vec{b}$ are unique iff

  i.e. only one sol'n

- If $A_{n\times n}$ is a square matrix

  then $\text{rref}(A) = I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} \Longrightarrow$

  $\text{rref}(A) \neq I \Longrightarrow$

Exercise 1 : Consider the 2nd order DE for $y(x)$:

$$y'' + 4y' + 3y = 6$$

1a) Show $y(x) = c_1 e^{-x} + c_2 e^{-3x} + 2$ solves this DE

1b) Find values for c_1, c_2 to solve the IVP for this DE, with

$$y(0) = 4$$
$$y'(0) = 1$$

1c) What if the initial values were $y(0) = y_0$, $y'(0) = v_0$?

(just to indicate breadth of possible applications)

2) Balance the following combustion equation. Methane, CH_4, is the primary component of natural gas, and we've all heard about the dangers of carbon monoxide which are created if there is not enough O_2.

$$a \cdot CH_4 + b \cdot O_2 \longrightarrow c \cdot CO_2 + d \cdot H_2O + e \cdot CO$$

2a) Show
$$a = c + e$$
$$4a = 2d$$
$$2b = 2c + d + e$$

2b) Order the variables c, d, e, a, b & write the linear system in synthetic form. Compute reduced row echelon form to solve for c, d, e in terms of a, b.

2c) Discuss implications!

soln: $a\,CH_4 + b\,O_2 \longrightarrow (-3a+2b)\,CO_2 + (2a)\,H_2O + (4a-2b)\,CO$

We've been noticing a relationship between solutions of homogeneous matrix eqtns and non-homogeneous ones. This actually just depends on $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ and $A(c\vec{x}) = cA(x)$:-

Theorem The general solution to

$$A\vec{x} = \vec{b}$$

is of the form $\vec{x} = \vec{x}_p + \vec{x}_H$

$\vec{x}_p$: (any) particular solution

$\vec{x}_H$: general solution to the homogeneous equation $A\vec{x} = \vec{0}$

proof: If $A\vec{x}_p = \vec{b}$ and $A\vec{x}_H = \vec{0}$

then $A(\vec{x}_p + \vec{x}_H) = A\vec{x}_p + A\vec{x}_H$
$= \vec{b} + \vec{0} = \vec{b}$

If $\vec{x}$ is any soltn to

$$A\vec{x} = \vec{b}$$

then
$$\vec{x} = \vec{x}_p + (\vec{x} - \vec{x}_p)$$

so $A\vec{x} = A[\vec{x}_p + (\vec{x} - \vec{x}_p)]$
$= A\vec{x}_p + A(\vec{x} - \vec{x}_p)$

so $\vec{b} = \vec{b} + A(\vec{x} - \vec{x}_p)$

so $\vec{0} = A(\vec{x} - \vec{x}_p)$

so $\vec{x} - \vec{x}_p$ is a sol'n to the homog. eqtn, i.e.

$\vec{x} - \vec{x}_p = \vec{x}_H$

∎

"same" as

Let $L(y) := y' + p(x)y$

$L(y_1 + y_2) = L(y_1) + L(y_2)$

$L(cy) = cL(y)$

Check!

so same conclusion holds, to solve $L(y) = q(x)$,

i.e. $y' + p(x)y = q(x)$

gen'l sol'n is $y = y_P + y_H$!

example 1

$$y' + 3y = 6$$
$$e^{3x}(y' + 3y) = 6e^{3x}$$
$$(e^{3x}y)' = 6e^{3x}$$
$$e^{3x}y = 2e^{3x} + C$$
$$y = \underbrace{2}_{y_P} + \underbrace{Ce^{-3x}}_{y_H}$$

!!

example 2

Exercise 1, page 3 today.