Math 2250-1
Wed 9/14

§3.1-3.2 linear systems of equations

In these sections we will learn how to find all solutions $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ to linear systems of equations:

$$\text{L.S.} \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m . \end{array} \right.$$

as well as the structure of the solution space. (Here the coefficients a_{ij} are known, ~~the~~ b_i are known, and we seek all soltns $\vec{x}$.)

Let's start small!

1) Describe the solution set to each equation below, and sketch its geometric realization

a) $3x = 5$
$x \in \mathbb{R}$

b) $2x + 3y = 6$
$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

c) $2x + 3y + 4z = 12$
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$

summary:

		geometric soltn set
1 linear equation in 1 unknown "x":	$ax = b$ →	
1 linear eqtn in 2 unknowns "x, y":	$ax + by = c$ →	
1 linear eqtn in 3 unknowns "x, y, z"	$ax + by + cz = d$	

2 linear equations in 2 unknowns :

$a_1 x + b_1 y = c_1$
$a_2 x + b_2 y = c_2$

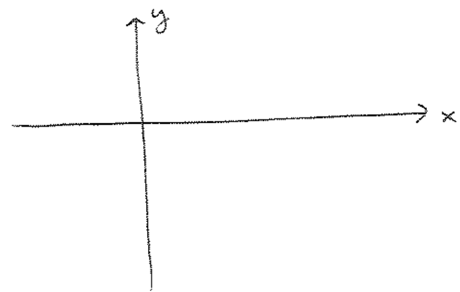
goal: find all $\begin{bmatrix} x \\ y \end{bmatrix}$ so that both eqtns hold

"simultaneous solution"

2) Consider

$E_1 \quad 5x + 3y = 1$
$E_2 \quad x - 2y = 8$

2a) Sketch the solution set (which is a single point in this case) geometrically as an intersection of two lines



2b) Use the following "elementary equation operations" to systematically reduce the system E_1, E_2 to

$1 \cdot x + 0 \cdot y = d_1$
$0 \cdot x + 1 \cdot y = d_2$

Make sketches at each stage, of the intersecting lines

- interchanging order of eqtns does not change solution set
- multiplying an eqtn by a non-zero constant does not change sol'n set
- replacing an eqtn with its sum with a multiple of another eqtn does not change soltn set to the system of eqtns! (Why?)

soltn $x = 2$, $y = -3$

2c) Look at your work on the previous page. Notice you could have saved a lot of time by doing this computation "synthetically", i.e. just keeping track of the coefficients and right hand sides.
Your work might look like

$$
\begin{array}{r|rr|r}
 & 5 & 3 & 1 \\
 & 1 & -2 & 8 \\
\hline
R_2 & 1 & -2 & 8 \\
R_1 & 5 & 3 & 1 \\
\hline
 & 1 & -2 & 8 \\
-5R_1+R_2 & 0 & 13 & -39 \\
\hline
 & 1 & -2 & 8 \\
 & 0 & 1 & -3 \\
\hline
2R_2+R_1 & 1 & 0 & 2 \\
 & 0 & 1 & -3
\end{array}
\longrightarrow
\begin{array}{l} x = 2 \\ y = -3 \end{array} \;!
$$

So, what are the possible solution sets to 1, 2, 3, 4, or any number of linear equations in 2 unknowns?

3) <u>example</u>: Consider the 2nd order I.V.P.

$$\text{IVP} \begin{cases} y'' - 9y = 0 \\ y(0) = 7 \\ y'(0) = 9 \end{cases}$$

- Show $y(x) = Ae^{3x} + Be^{-3x}$ is always a sol'n (A, B consts)

$y' =$

$y'' =$

$y'' - 9y =$

- Solve the IVP: (you'll need to solve a linear system of 2 equations to get A, B.)

ans $y = 5e^{3x} + 2e^{-3x}$

simultaneous sol'ns to linear eqtns in 3 unknowns: geometric meaning?

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 9z &= 23 \end{aligned}$$

$$\begin{array}{llcccc|c} & & 1 & 2 & 1 & & 4 \\ & & 3 & 8 & 7 & & 20 \\ & & 2 & 7 & 9 & & 23 \end{array}$$

$$\begin{array}{lccc|c} & 1 & 2 & 1 & 4 \\ -3R_1+R_2 & 0 & 2 & 4 & 8 \\ -2R_1+R_2 & 0 & 3 & 7 & 15 \end{array}$$

$$\begin{array}{lccc|c} & 1 & 2 & 1 & 4 \\ R_2/2 & 0 & 1 & 2 & 4 \\ & 0 & 3 & 7 & 15 \end{array}$$

$$\begin{array}{lccc|c} & 1 & 2 & 1 & 4 \\ & 0 & 1 & 2 & 4 \\ -3R_1+R_2 & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{array}{lccc|c} -R_3+R_1 & 1 & 2 & 0 & 1 \\ -2R_3+R_2 & 0 & 1 & 0 & -2 \\ & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{array}{lccc|c} -2R_2+R_1 & 1 & 0 & 0 & 5 \\ & 0 & 1 & 0 & -2 \\ & 0 & 0 & 1 & 3 \end{array}$$

so $x = 5$, $y = -2$, $z = 3$

check ans!

$$\begin{array}{ll} & x + 2y + z = 4 \\ -3E_1+E_2 & 0 + 2y + 4z = 8 \\ -2E_1+E_2 & 0 + 3y + 7z = 15 \end{array}$$

$$\begin{array}{ll} & x + 2y + z = 4 \\ E_2/2 & \quad y + 2z = 4 \\ & \quad 3y + 7z = 15 \end{array}$$

$$\begin{array}{ll} & x + 2y + z = 4 \\ & \quad y + 2z = 4 \\ -3E_2+E_3 & \qquad z = 3 \end{array}$$

$\leftarrow z = 3$

so $y = 4 - 6 = -2$

so $x = 4 - 3 + 4 = 5$

$$\begin{array}{ll} \text{or continue } -E_3+E_1, \ -2E_3+E_2 & x + 2y = 1 \\ & \quad y = -2 \\ & \quad z = 3 \end{array}$$

$$\begin{array}{ll} -2E_1+E_2 & x = 5 \\ & y = -2 \\ & z = 3 \end{array}$$

geometric picture?

other possibilities! (Almost same system as page 5)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= \begin{cases} 20 \\ 23 \end{cases} \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 8 & \begin{cases} 20 \\ 23 \end{cases} \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1+R_2 \\ -2R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 6 & \begin{cases} 12 \\ 15 \end{cases} \end{array}\right]$$

$$\begin{array}{r} \\ R_2/2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 6 & 12 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_3/3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} -2R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$ (arbitrary)
$y = 4 - 2t$
$x = -4 + 3t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4+3t \\ 4-2t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Line of sol'ns!

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 6 & 15 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -3R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

?? $0 = -1$

NO SOL'N