Math 2250-1
Fri 10/7 Finish 4.1-4.4

Finish Wed. notes, p. 2-3 discuss. pages 1-2 in today's notes [p. 3-5 are a summary of the rest of the past week, included in case you might find the summary helpful.]

Related to exercise 17, it is useful to know that if

$\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ are linearly independent vectors in V
but they don't span V,
then if you pick any $\vec{w} \in V$, $\vec{w} \notin span\{\vec{v}_1, \ldots \vec{v}_k\}$
the larger collection

$\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k, \vec{w}$ will be linearly independent.

check: $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k + d\vec{w} = \vec{0}$

if $d = 0$, then $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$
so $c_1 = c_2 = \ldots = c_k = 0$ also.

and we can't have $d \neq 0$, since
then $\vec{w} = -\frac{c_1}{d}\vec{v}_1 - \frac{c_2}{d}\vec{v}_2 - \ldots - \frac{c_k}{d}\vec{v}_k$

and $\vec{w} \notin span\{\vec{v}_1, \ldots \vec{v}_k\}$. ∎

in this way you can successively build up to a basis for V, containing the original independent vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$.

the explanation of the facts about dimension & basis on page 3 of Wed. notes is on the next page of today's notes.

Key facts about independence, span, basis, dimension (§4.4)

(in a fixed vector space V)

(1) Primary fact : If a finite collection of vectors spans a vector space V, then any collection having a greater number of vectors is dependent

reason : Let $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ span V, let $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_{k+\ell}\}$ $(\ell > 0)$ be a larger collection

We search for lin. comb. coeff's c_j s.t.

$$c_1\vec{w}_1 + c_2\vec{w}_2 + \dots + c_{k+\ell}\vec{w}_{k+\ell} = \vec{0}$$

Express each $\vec{w}_j$ as a linear combo of the spanning set:

$$c_1 \begin{bmatrix} a_{11}\vec{v}_1 \\ +a_{21}\vec{v}_2 \\ + \\ \vdots \\ +a_{k1}\vec{v}_k \end{bmatrix} + c_2 \begin{bmatrix} a_{12}\vec{v}_1 \\ +a_{22}\vec{v}_2 \\ \vdots \\ +a_{k2}\vec{v}_k \end{bmatrix} + \dots + c_{k+\ell} \begin{bmatrix} a_{1\,k+\ell}\vec{v}_1 \\ +a_{2\,k+\ell}\vec{v}_2 \\ \vdots \\ +a_{k\,k+\ell}\vec{v}_k \end{bmatrix} = \begin{bmatrix} 0\vec{v}_1 \\ +0\vec{v}_2 \\ + \vdots \\ +0\vec{v}_k \end{bmatrix}$$

equating coeff's for each $\vec{v}_i$, we get a dependency if we can find a $\vec{c} \neq \vec{0}$ so that

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \\ \vec{c} \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{0} \end{bmatrix}$$

But the solution space to this homogeneous equation is at least ℓ-dimensional, so non-zero sol'ns $\vec{c}$ exist. ∎

Logical consequences of (1):

(2) If a finite collection of vectors in V is independent, no collection with fewer vectors can span

reason : logic! If a smaller set did span, the larger set would've been dependent! ∎

(3) If $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is a basis for V, then every other basis also consists of exactly k vectors

reason : fewer vectors can't span, more vectors would be dependent. ∎

(4) If $\dim V = k$ and if $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_k\}$ are independent, they span! (and are a basis)

reason : if they didn't span there would be a $\vec{v} \in V$ not in their span, so that the set $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_k, \vec{v}\}$ would still be independent. This would violate (1)! ∎

(5) If $\dim V = k$ and if $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_k\}$ span, then they're independent! (so are a basis)

reason : if they were dependent we could throw one of them away (i.e. a dependent one) without shrinking the span, so we would have $(k-1)$ vectors spanning V, in violation of (2)! ∎

4.1 - 4.4 summary
(for reference)

(also include the facts about dimension & bases on the previous page.)

Since last Friday (and in HW) we've been studying problems and concepts related to

<u>linear combinations</u> $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$ of vectors $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$.

if we're working in $\mathbb{R}^m$ we answer all such questions using the matrix theory from Chapter 3, because

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = m\left\{\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix}}_{k}\right. \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = B\vec{c}$$

<u>linear independence / dependence</u> for $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$:

independence: $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0} \Rightarrow c_1 = c_2 = \dots = c_k = 0$

dependence: $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$ for some choice of $c_1, c_2, \dots c_k$ with not all of $c_1, c_2, \dots c_k = 0$.

test:

$$B\vec{c} = \vec{0} \longrightarrow B \mid \vec{0}$$

$$\downarrow$$

$$\text{rref}(B) \mid \vec{0}$$

if every column of rref(B) has a leading 1, deduce $\vec{c} = \vec{0}$ $\boxed{\text{Independent}}$

if some column of rref(B) does not have a leading 1 $\Rightarrow$ free parameters $\Rightarrow$ ∞'ly many dependencies $\Rightarrow$ $\boxed{\text{dependent}}$

$$m \begin{bmatrix} \quad B \quad \end{bmatrix}_{k}$$

this is guaranteed to happen if the number of vectors k, is greater than the m in $\mathbb{R}^m$

<u>span</u> of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ is the collection of all their linear combinations:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\} = \{\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k, \quad c_j \in \mathbb{R} \quad 1 \le j \le k\}.$$

We checked that the linear combination coefficients for each $\vec{w} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ are unique if and only if $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ are linearly independent. This is the value of lin. ind. vectors – there is no redundancy in how we express vectors in their span, using linear combinations.

Do $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ span $\mathbb{R}^m$: solve $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{x}$ for $c_1, c_2, \ldots c_k$, whenever $\vec{x}$ is given.

$$B\vec{c} = \vec{x}$$ solve for $\vec{c}$!

test: $B \,\vdots\, \vec{x}$

$$\downarrow$$

$\text{rref}(B) \,\vdots\, \vec{z}$

if rref(B) has no zero rows, backsolve to find $\vec{c}$'s, regardless of $\vec{z}$, so $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k$ span $\mathbb{R}^m$

if rref(B) has a row of zeroes, then there are conditions $\vec{x}$ must satisfy (to make the z_j's $= 0$ in each row$_j$ of rref(B) that $= \vec{0}$) so that $\vec{x}$ is in the span. Thus $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ do NOT span $\mathbb{R}^m$

$$\boxed{\text{this is guaranteed to happen if the number of vectors } k \text{ is less than the } m \text{ in } \mathbb{R}^m.}$$

$$\begin{array}{cc} & k \\ m & \begin{bmatrix} B \\ \\ \end{bmatrix} \end{array}$$

a <u>subspace</u> W of a <u>vector space</u> V is a special subset of V that is

closed under addition (α) and

closed under scalar multiplication (β).

This means that W is itself a vector space.

The two ways we have seen subspaces arise are

1) $W = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$

2) W = solution space to $A_{m\times n}\vec{x} = \vec{0}$, i.e. $\{\vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}\}$.

a <u>basis</u> of a vector space V (which could be a subspace of a larger vector space) is a linearly independent collection of vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$

1) that span V and

2) are linearly independent.

the <u>dimension</u> of V is the number of vectors in a basis for V

in other contexts, dimension may also be called "degrees of freedom". It is the number of scalar parameters needed to describe all elements of V in a non-redundant (i.e. unique) way.

If $W = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$

eliminate dependent vectors to create a basis

in $\mathbb{R}^m$:

$$\left[\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{array}\;\middle|\;\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array}\right]$$

$$\Big\downarrow \text{rref}$$

$$\left[\begin{array}{c|c|c|c} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_k \end{array}\;\middle|\;\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right]$$

read dependencies for $\vec{v}_1, \ldots \vec{v}_k$ from those for $\vec{u}_1 \ldots \vec{u}_k$.

If W = solution space to $A\vec{x} = \vec{0}$

$$A \,\vdots\, \vec{0}$$

$$\Big\downarrow \text{rref}$$

$$\text{rref}(A) \,\vdots\, \vec{0}$$

backsolve;

put in linear combination form

$$\vec{x} = t_1\vec{w}_1 + \cdots + t_\ell \vec{w}_\ell \qquad t_1, t_2 \ldots t_\ell \in \mathbb{R}$$

$\vec{w}_1, \vec{w}_2, \ldots \vec{w}_\ell$ will always be a basis for the solution space.