Math 2250
Mond. 10/31

- variation of parameters from §5.5
- forced oscillations in mechanical systems §5.6
  (we'll finish §5.6 tomorrow.)

Variation of parameters: there is a formula for finding a particular solution $y_p(x)$ to any n^{th} order non-homogeneous linear DE for $y(x)$,

$$L(y) := y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = f(x)$$

This method is more complicated to use in cases like we considered Friday using "guessing", i.e. the method of undetermined coefficients.
But → it works even if $f(x)$ is not of one of the special forms required for undetermined coef's, and even if the coeff functions $p_{n-1}, \ldots p_0$ are not constants.

Here's the formula: Let $y_1(x), y_2(x), \ldots y_n(x)$ be a basis to the homogeneous solutions, i.e. for sol'ns to $L(y)=0$.

Then $y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) + \cdots + u_n(x) y_n(x)$ solves $L(y_p) = f$,

where $u_1(x), u_2(x), \ldots u_n(x)$ are chosen so that

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \left[W(y_1, y_2, \ldots y_n) \right]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

Wronskian matrix inverse.

Check for $n=2$:

$$L(y) = y'' + p(x) y' + q(x) y$$

try $\Downarrow$ $\quad y_H = c_1 y_1(x) + c_2 y_2(x)$

$$q(x)\,(\,y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$
$$p(x)\,(\,y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2 \quad \text{① set} = 0$$
$$1\,(\,y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' \quad \text{② set} = f$$

$$\Rightarrow L(y_p) = u_1 (\underbrace{L(y_1)}_{0}) + u_2 \underbrace{L(y_2)}_{0} \quad + f$$

$$L(y_p) = f$$

①,② in matrix form:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

i.e.

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = [W]^{-1} \begin{bmatrix} 0 \\ f \end{bmatrix} \quad \checkmark$$

for $n=2$ this simplifies:
$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix} = \frac{1}{W} \begin{bmatrix} -f y_2 \\ f y_1 \end{bmatrix}$$

§5.6 Forced oscillations

Overview

$$m x'' + c x' + k x = F_0 \cos\omega t$$

- undamped $(c=0)$

  $$m x'' + k x = F_0 \cos\omega t$$

  - $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ $\quad x_p = A\cos\omega t + B\sin\omega t$
    $\quad = C\cos(\omega t - \alpha)$

    general soltn
    $x = x_p + x_H$
    $\quad = C\cos(\omega t - \alpha)$
    $\quad + C_0 \cos(\omega_0 t - \alpha_0)$

  - $\omega = \omega_0$ $\quad x_p = t(A\cos\omega_0 t + B\sin\omega_0 t)$
    $\quad = t\, C\cos(\omega_0 t - \alpha)$ $\qquad$ general soltn $x = x_p + x_H$

    RESONANCE!

  - $\omega \neq \omega_0$, but ω close to ω_0
    BEATING

- damped $(c > 0)$ (More details Tuesday)

  $x_p = A\cos\omega t + B\sin\omega t$
  $\quad = C\cos(\omega t - \alpha)$

  $x_H \rightarrow 3$ possibilities (over, critically, underdamped)
  key feature they all share is $x_H(t) \to 0$ exponentially fast as $t \to \infty$.

so x_p is called x_{sp} (steady periodic)
x_H is called x_{tr} (transient)

when damping c is small, and ω is near ω_0, then amplitude C of x_{sp} will (often) be a large multiple of forcing amplitude F_0. This is called practical resonance
(or, if you deal in electronics, you might call it resonance.)

Exercise 1 Solve IVP: $m=1$, $k=9$, $F_0=80$, $\omega=5$, $\omega_0=?$

$$\begin{cases} x''(t) + 9x(t) = 80\cos 5t \\ x(0)=0 \\ x'(0)=0 \end{cases}$$

x_H:

x_p:

then solve IVP.
discuss periodicity

ans

$$x(t) = \underbrace{5\cos 3t}_{\substack{\text{period} \\ T_0=\frac{2\pi}{3}}} - \underbrace{5\cos 5t}_{\substack{\text{period} \\ T_1=\frac{2\pi}{5}}}$$

(and all integer multiples).

the period of the sum will be the least common integer multiple of T_0 and T_1, which in this case is

1b) $\boxed{}$

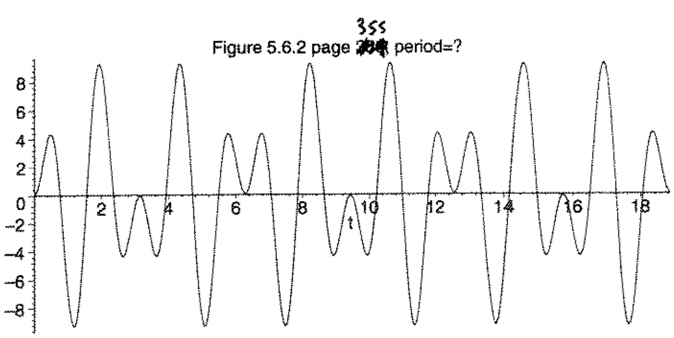
1c) for $x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$

$x(t) = x_p + x_H$.

When will the sum be periodic, when won't it be?

Example 1 p. 354

```
> with(plots):
> plot(5*cos(3*t)-5*cos(5*t),t=0..6*Pi,color=black,
  title='Figure 5.6.2 page 350, period=?');
```



undamped forced IVP, $\omega \neq \omega_0$, with letters

$$\begin{cases} x'' + \frac{k}{m} x = \frac{F_0}{m} \cos\omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\begin{aligned}
+\tfrac{k}{m}\ (\ x_p &= A\cos\omega t\ ) \\
+0\ (\ x_p' &= -A\omega \sin\omega t\ ) \\
+1\ (\ x_p'' &= -A\omega^2 \cos\omega t\ ) \\
\hline
L(x_p) &= \cos\omega t\ A\left[\underbrace{\tfrac{k}{m}}_{\omega_0^2} - \omega^2\right]
\end{aligned}$$

deduce $A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$

$$A = \frac{F_0}{m}/(\omega_0^2-\omega^2)$$

so,

$x_p(t) = -\frac{F_0}{m}\frac{1}{(\omega^2-\omega_0^2)}\cos\omega t$. Note $x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t$.

so, by plugging in or observation IVP solution is

$$\boxed{x(t) = \frac{+F_0}{m(\omega^2-\omega_0^2)}\left(\cos\omega_0 t - \cos\omega t\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

check IVP!

when ω is near (but $\neq 0$) ω_0, this sum varies between ± 2, depending on whether the two terms are in, or out of phase....
trig makes this precise!!

$$\begin{aligned}
&\cos(\alpha-\beta) - \cos(\alpha+\beta) \\
&\quad \| \\
&\cancel{\cos\alpha\cos\beta} + \sin\alpha\sin\beta - (\cancel{\cos\alpha\cos\beta} - \sin\alpha\sin\beta) \\
&= 2\sin\alpha\sin\beta
\end{aligned}$$

let $\left.\begin{aligned} \alpha &= \bar{\omega} t = \left(\tfrac{\omega+\omega_0}{2}\right)t \\ \beta &= \delta t = \left(\tfrac{\omega-\omega_0}{2}\right)t \end{aligned}\right\}$ so $\begin{aligned} \alpha-\beta &= \omega_0 t \\ \alpha+\beta &= \omega t \end{aligned}$

i.e.

$$\boxed{x(t) = \frac{F_0}{m(\omega^2-\omega_0^2)}\cdot 2\sin\bar{\omega}t\ \sin\delta t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

period $\frac{2\pi}{\bar{\omega}}$, $\bar{\omega} = \frac{\omega+\omega_0}{2}$

period $\frac{2\pi}{\delta}$ is huge if $\omega \approx \omega_0$; $\delta = \frac{\omega-\omega_0}{2}$ $\boxed{\text{Beating}}$

Notice the beating amplitude $\frac{2F_0}{m(\omega^2-\omega_0^2)}$ blows up as $\omega \to \omega_0$

Exercise 2:

Keep the same data as in #1 : $m=1$, $k=9$, $F_0=80$
$\omega_0 = 3$
except choose $\omega = 3.1$

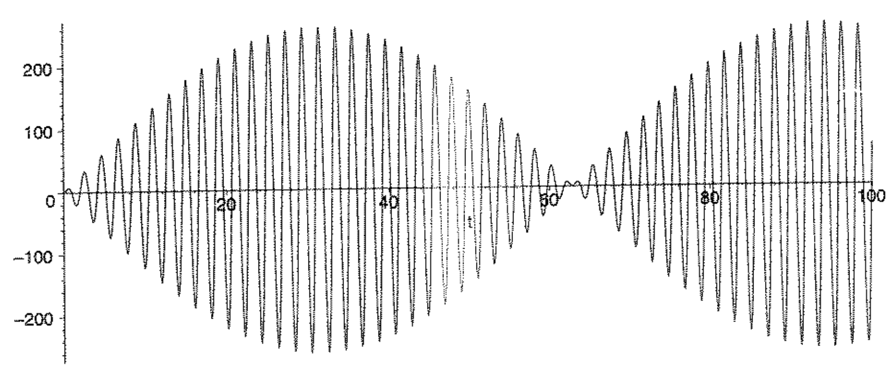
Use second box to solve IVP

$$\begin{cases} x'' + 9x = 80\cos(3.1t) \\ x(0)=0 \\ v(0)=0 \end{cases}$$

and compute the beating period and amplitude

ans $x(t) \cong 262.3 \sin(3.05t)\sin(.05t)$

$T_{beat} \approx 126$ (actually ≈ 63)



Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned}
+\omega_0^2\,(\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \quad) \\
0\,(\quad & x_p' = t\,(-A\omega_0 \sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \quad) \\
+1\,(\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2 \sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \quad)
\end{aligned}$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$
$B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\, t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

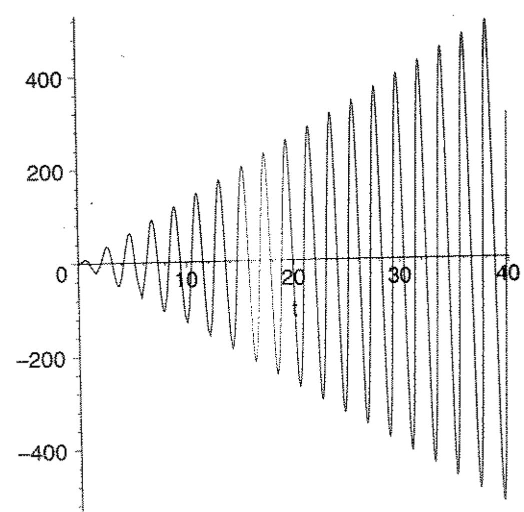
(you can also guess this by letting $\omega \to \omega_0$ on page 3, second box, via linearization!)

No matter how small $F_0 \neq 0$, soln blowup as $t \to \infty$!

<u>Exercise 3</u>, as before $m=1$, $k=9$ $(\omega_0 = 3)$
$F_0 = 80$
$\omega = 3 = \omega_0$

$$\begin{cases} x'' + 9x = 80\cos 3t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$x(t) = ?$



ans $x(t) = \frac{40}{3}\, t\sin 3t$

Exercise 4

$x_p(t)$ for resonance on page 6, via variation of parameters instead of undetermined coefficients.

$$x'' + \omega_0^2 x = \cos\omega_0 t$$

$x_H(t)$:
$x_1(t) = \cos\omega_0 t$, $x_2(t) = \sin\omega_0 t$

$$[W] = \begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\omega_0 \sin\omega_0 t & \omega_0 \cos\omega_0 t \end{bmatrix}$$

$$x_p(t) = u_1(t)\, x_1(t) + u_2(t)\, x_2(t)$$

$$\det[W] = W = \omega_0$$

from page 1, converting $y = y(x)$ to $x = x(t)$:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W}\begin{bmatrix} -fy_2 \\ fy_1 \end{bmatrix} = \frac{1}{\omega_0}\begin{bmatrix} -\cos(\omega_0 t)\sin(\omega_0 t) \\ \cos(\omega_0 t)\cos(\omega_0 t) \end{bmatrix} \quad \ldots$$

ans possibly

$$u_1 = \frac{1}{2\omega_0^2}\sin^2\omega_0 t$$

$$u_2 = \frac{t}{2\omega_0} + \frac{1}{4\omega_0^2}\sin 2\omega_0 t$$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{t}{2\omega_0}\sin(\omega_0 t)$$