Math 2250-1
Monday 10/3 §4.1-4.3

Vocab review!

If $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ are vectors, then

a linear combination $\vec{w}$ of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ is?

the span of $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is?

In our examples Friday, when we considered vectors $\vec{b}$ in span$\{\vec{v}_1, \ldots \vec{v}_k\}$, sometimes the linear combination coefficients were unique, and in examples with different collections of vectors $\{\vec{w}_1, \ldots \vec{w}_k\}$, they weren't.

This is related to precise defintion of linear independence vs. dependence:

Definition

a) $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ are linearly independent iff the only linear combination

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$$

is the one where $c_1 = c_2 = \ldots = c_k = 0$.

b) $\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k$ are linearly dependent iff some linear combination

$$c_1\vec{w}_1 + c_2\vec{w}_2 + \ldots + c_k\vec{w}_k = \vec{0}$$

where not all of the c_j's are zero.

- an equivalent way to say $\vec{w}_1, \ldots \vec{w}_k$ are lin. dep. is to say that at least one of the vectors $\vec{w}_j$ $(1 \le j \le k)$ can be expressed as a linear combination of the other vectors, i.e. is "linearly dependent" on them

an equivalent way to say $\vec{v}_1, \ldots \vec{v}_k$ are lin. ind. is to say that none of the $\vec{v}_j$ can be expressed as a linear comb of the rest.

- linear independence is a good condition for a collection of vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ to have because it means that if

$$\vec{b} = r_1\vec{v}_1 + r_2\vec{v}_2 + \ldots + r_k\vec{v}_k$$

and also $\vec{b} = s_1\vec{v}_1 + s_2\vec{v}_2 + \ldots + s_k\vec{v}_k$ then actually all $r_j = s_j$, i.e. linear combo coeffs are unique.

(because $\vec{b} - \vec{b} = \vec{0} = (r_1 - s_1)\vec{v}_1 + (r_2 - s_2)\vec{v}_2 + \ldots + (r_k - s_k)\vec{v}_k$

so $r_1 - s_1 = 0,\ r_2 - s_2 = 0, \ldots r_k - s_k = 0$.)

Finish Exercise 3 Friday

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$

2) b)
$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 6 \\ 1 & 1 & -1 & 3 \\ 1 & -5 & 11 & 3 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so
$$\underbrace{c_1\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} + c_3\begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}}_{\begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & -1 \\ 1 & -5 & 11 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

for
$c_1 = 3 - t$
$c_2 = 2t$
$c_3 = t$

so $\vec{b} \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ but linear combo coeff's <u>not</u> unique

thus, by our reasoning on page 1,
$\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be linearly dependent.

c) modified.
- Find a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ which adds up to $\vec{0}$.

note
$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & -1 \\ 1 & -5 & 11 \end{bmatrix}\begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

d) Your work in (c) shows
$$\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\{\vec{v}_1, \vec{v}_2\}.$$

find the <u>implicit equation</u> of the plane spanned by $\vec{v}_1, \vec{v}_2$.

Hint: for what $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ can we solve $c_1\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. ?

Exercise 4a) If you have 3 vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$, what matrix condition is equivalent to

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3 \ ?$$

4b) What matrix condition is equivalent to $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent?

4c) What determinant condition tests both of these?

$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$ are examples of Vector spaces. Such spaces allow for addition and scalar multiplication, and some authors call them linear combination spaces. Here is the precise definition:

A set V of "vectors", together with operations $+$, scalar multiplication is called a vector space if the following axioms hold

(α) whenever $\vec{u}, \vec{v} \in V$ then $\vec{u} + \vec{v} \in V$ (closure wrt addition)

(β) whenever $\vec{u} \in V$, $k \in \mathbb{R}$, then $k\vec{u} \in V$ (" " scalar multiplication)

(a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ commutative

(b) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ associative

(c) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ Zero vector exists in V

(d) $\vec{u} + (-\vec{u}) = \vec{0}$ additive inverses exist

(e) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ distributive prop

(f) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$ "

(g) $a(b\vec{u}) = (ab)\vec{u}$

(h) $1\vec{u} = \vec{u}$, $(-1)\vec{u} = -\vec{u}$, $0\vec{u} = \vec{0}$ (these last 2 actually follow from the others) any subset W of a vectorspace V that is "closed" under addition and scalar multiplication, i.e.

α) whenever $\vec{u}, \vec{v} \in W$ then $\vec{u}+\vec{v} \in W$

β) whenever $\vec{u} \in W$, $k \in \mathbb{R}$, then $k\vec{u} \in W$

inherits properties (a)–(h) from the vector space V so <u>is itself a vector space</u>.

Thus, W is called a <u>subspace</u> of V if it is closed under addition and scalar multipl.

<u>Exercise 5</u> : Consider span$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} \right\}$ as a subset of $\mathbb{R}^3$.

is this <u>subset</u> a <u>subspace</u> of $\mathbb{R}^3$?

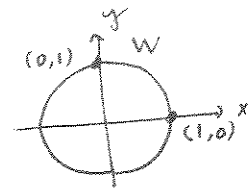
<u>Exercise 6</u> In exercise 3, we showed span$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} \right\}$ is the collection of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

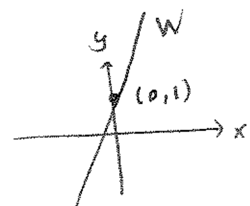
which satisfy $\boxed{-2x + 3y + z = 0.}$

Verify that the points on this plane are a <u>subspace</u> of $\mathbb{R}^3$ using this implicit representation

Exercise 7

Is $W = \{(x,y) \in \mathbb{R}^2 \text{ s.t. } x^2+y^2=1\}$ a subspace of $\mathbb{R}^2$?





Exercise 8 Is $W = \{(x,y) \in \mathbb{R}^2 \text{ s.t. } y = 3x+1\}$ a subspace of $\mathbb{R}^2$?

Exercise 9 What are the only 3 kinds of subspaces of $\mathbb{R}^2$?

just what are the possible subspaces of $\mathbb{R}^2$? $\mathbb{R}^3$? $\mathbb{R}^4$?