Math 2250-1
Fri 10/28

§5.5 Finding y_P for $L(y)=f$.

Recall, if L is a linear operator $\left(\begin{array}{l} L(y_1+y_2)=L(y_1)+L(y_2) \\ L(cy_1)=cL(y_1) \end{array} \right)$

then the general sol'n to

$$L(y)=f$$

is $y=y_P+y_H$

where y_P is any single particular sol'n and
y_H is the general solution to the homogeneous problem $L(y)=0$.

How to find y_P for

$$L(y)=a_n y^{(n)}+a_{n-1}y^{(n-1)}+\dots+a_1 y'+a_0 y=f$$

There are two methods:

- undetermined coefficients → works for $f(x)$ of certain types
  success guaranteed by linear algebra
- variation of parameters → always works, regardless of $f(x)$.
  disadvantage is - the formulas are messy.

next week, when we have time.

most of our focus is on this method.

Exercise 1 Find a particular sol'n y_P to the DE

$$y''+4y'-5y=10x+3$$

hint: try $y_P=Ax+B$

<u>Exercise 2</u> What is the general sol'n to

$$y'' + 4y' - 5y = 10x + 3 \; ?$$

<u>Exercise 3</u> Find a particular sol'n to

$$y'' + 4y' - 5y = 14e^{2x}$$

hint: try $y_p = Ae^{2x}$

<u>Exercise 3b</u> y_p for

$$y'' + 4y' - 5y = 14e^{2x} + 20x + 6$$

Hint: use Ex 1, 3 & superposition

Exercise 4 Find a particular sol'n to

$$y'' + 4y' - 5y = 2\cos 3x$$

$$y_p = ?$$

Hint: to solve $L(y) = f$
for a particular sol'n y_p,
we hope f is in a finite dimensional
subspace preserved by L, i.e.
$L : V \to V$ with $f \in V$.

In exercise 1, $V = \text{span}\{1, x\}$
In exercise 3, $V = \text{span}\{e^{2x}\}$.
What should V be in exercise 4?

Method of undetermined coefficients

If f is in a finite dimensional subspace V, with $L: V \to V$, and if the only function in V with $L(y)=0$ is $y=0$, then there is a unique $y_p \in V$ with $L(y_p) = f$

(This is because of the rank + nullity theorem, applied to $L: V \to V$)

$$\underbrace{\dim(\text{nullspace}(L))}_{0} + \dim(\text{range}(L)) = \dim(V)$$

So, if L is a constant coefficient linear differential operator, what type of y_p would you try in the following problems?

$L(y) = x^3 + 6x^2 - 5$

$y_p =$

$L(y) = 4e^{2x}\sin 3x$

$y_p =$

$L(y) = 7e^{13x}$

$y_p =$

$L(y) = x \sin 2x$

$y_p =$

Exercise 5 Find y_p for

$$y'' + 4y' - 5y = 4e^x$$

notice that $y_H = c_1 e^x + c_2 e^{-5x}$

so $y_p = Ae^x$ won't get you very far

try $y_p = Axe^x$!!

See discussion p 341-346. Also p. 346 Table: Don't forget to also use superposition for $f_1 + f_2$

$f(x)$	y_p	~~potential~~ related root of $p(r)$
$P_m(x) = b_0 + b_1 x + \dots + b_m x^m$	$x^s(A_0 + A_1 x + A_2 x^2 + \dots + A_m x^m)$	$r = 0$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$	$r = ik$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$	root $= r + ik$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1 x + \dots + A_m x^m)e^{rx}$	root $= r$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s\left[(A_0 + A_1 x + \dots + A_m x^m)\cos kx + (B_0 + B_1 x + \dots + B_m x^m)\sin kx\right]$	root $= ik$

$x^s = 1$ $(s = 0)$ as long as the potential root is not actually a root of the characteristic polynomial for L (easy case, Exercises 1–4)

Otherwise, s is the power of $(r - \text{root})$ in factored $p(r)$

e.g. in Exercise 5,

$$p(r) = (r+5)^1 (r-1)^1$$

so multiply guess by x^1.