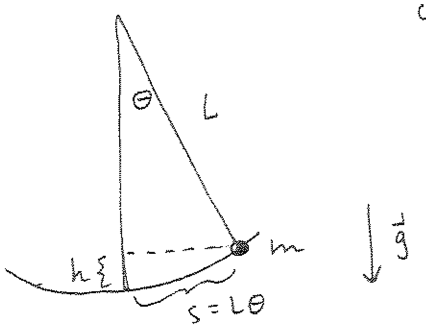
Math 2250-1
Wed 10/26
§5.4

- Do the example IVP's on page 5 of Tuesday's notes, illustrating
  - overdamped
  - critically damped
  - underdamped

Then, let's experiment!

1) pendulum



$\theta = \theta(t)$

conservative system $KE + PE = \text{const.}$

$$\frac{1}{2}mv^2 + mgh = \text{const}$$

$$s = L\theta$$

$$v = \frac{ds}{dt} = L\theta'(t) \qquad h = L - L\cos\theta = L(1-\cos\theta)$$

So, $$\frac{1}{2}m\,L^2(\theta'(t))^2 + mgL\,(1-\cos(\theta(t))) \equiv \text{const}$$

D_t: $$mL^2\,\theta'\theta'' + mgL(\sin\theta)\,\theta' \equiv 0$$

$$\underbrace{mL\theta'}_{\neq 0 \text{ except at isolated times}}\,(L\theta'' + g\sin\theta) \equiv 0$$

$\leadsto$ deduce eqtn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

linearize

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

$$\omega_0 = \sqrt{\frac{g}{L}}$$

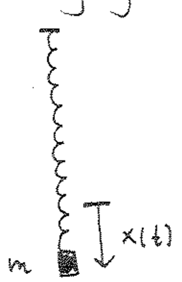
$$\theta(t) = C\cos(\omega_0 t - \alpha)$$

non-linear DE

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots$

$\sin\theta \approx \theta$ θ small is excellent approx (alternating series test)

2) hanging mass-spring:



$$mx'' = -kx$$

$$mx'' + kx = 0$$

$$\boxed{x'' + \frac{k}{m}x = 0}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Why don't you see gravity g in this DE?

Experiment notes
Math 2250-1
Wednesday October 26

Pendulum: measurements and prediction:

> $restart:$
> $Digits := 4:$

> $L := 1.526;$
> $g := 9.806;$
> $\omega := \mathrm{sqrt}(g/L);$ *# radians per second*
> $f := evalf(\omega/(2*\mathrm{Pi}));$ *# cycles per second*
> $T := 1/f;$ *# seconds per cycle*

$$L := 1.526$$
$$g := 9.806$$
$$\omega := 2.535$$
$$f := 0.4036$$
$$T := 2.478 \quad (1)$$

Experiment:

Mass–spring:
compute Hooke's constant:

> $104.0 - 88.3;$ *#displacement from extra 50g*

$$15.7 \quad (2)$$

> $k := \frac{.05 \cdot 9.806}{.157};$ *# solve k·x=m·g for k.*

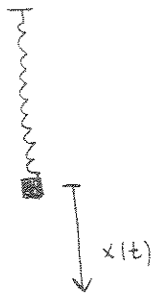
$$k := 3.123 \quad (3)$$

> $m := .1;$ *# mass for experiment is 100g*
> $\omega := \mathrm{sqrt}\left(\frac{k}{m}\right);$ *# predicted angular frequency*
> $f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right);$ *# predicted frequency*
> $T := \frac{1}{f};$ *# predicted period*

$$m := 0.1$$
$$\omega := 5.588$$
$$f := 0.8893$$
$$T := 1.124 \quad (4)$$

Experiment:

Improved spring model (account for spring mass)



normalize KE + PE = 0 for spring hanging at rest. (in gravity!)
then for system in motion

$$KE + PE = \frac{1}{2} m (x')^2 + KE_{spring} + \underbrace{\frac{1}{2} k x^2}_{\text{PE of system from equilibrium.}}$$

$$\left(= W = \int_0^x k x \, dx \right)$$

model: the top of the spring isn't moving. the bottom is moving with velocity $x'(t)$.
Assume the velocity of a piece of spring a fraction μ of the way $(0 \le \mu \le 1)$ from top to bottom is $\mu x'(t)$.
so $\text{speed}^2 = \mu^2 (x')^2$, for that piece. Thus

$$KE = \int_0^1 \frac{1}{2} \underbrace{\mu^2 (x'(t))^2}_{v^2} \underbrace{m_s \, d\mu}_{\text{mass}} = \frac{1}{6} m_s (x'(t))^2$$

So for $M := m + \frac{1}{3} m_s$,

$$KE + PE = \frac{1}{2} M (x')^2 + \frac{1}{2} k x^2 \equiv \text{const}$$

D_t:
$$M x' x'' + k x x' \equiv 0$$
$$x' (\underbrace{M x'' + k x}_{=0.}) \equiv 0$$
$$\omega_0 = \sqrt{\frac{k}{M}}$$

Does this improve our prediction?

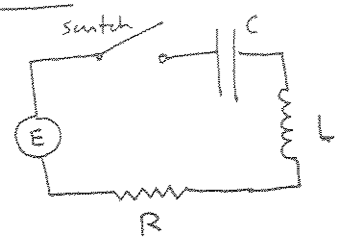
```
> ms := .011;  # spring has mass 11g
  M := m + ms/3;  # "effective mass"
```

$$ms := 0.011$$
$$M := 0.1037 \qquad (5)$$

```
> ω := sqrt(k/M); # predicted angular frequency
  f := evalf(ω/(2·Pi)); # predicted frequency
  T := 1/f; # predicted period
```

$$\omega := 5.488$$
$$f := 0.8734$$
$$T := 1.145 \qquad (6)$$

EP 3.7 (Electrical Circuits) [HW next week]



Voltage E volts (called Electromotive force, but really is energy/unit charge)

circuit elt	Voltage drop	units for component
Inductor	$L\frac{dI}{dt}$	L Henries H
Resistor	$R\,I(t)$	R ohms Ω
Capacitor	$\frac{1}{C}Q(t)$	C Farads F

$Q(t)$ = net charge on Capacitor (Coulombs)

$I(t) = Q'(t)$ = current (Amperes) (moving around loop).

(1 Volt = 1 Joule/Coulomb !)

Kirchoff's Law

the sum of the voltage drops around a circuit loop equals the applied voltage E.

(this says that a test particle traversing a closed loop returns with the same potential energy level it started at.)

$$\boxed{\begin{array}{l} L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = E(t) \\ D_t:\quad L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = E'(t) \end{array}}$$

mathematically identical to

$$\begin{array}{ccccccc} m\,x'' & + & c\,x' & + & k\,x & = & F(t) \\ \updownarrow & & \updownarrow & & \updownarrow & & \\ L & & R & & 1/C & & \end{array}$$

Exercise 1 : Set up an IVP for a circuit in which

$R = 16\ \Omega$
$L = 2\ H$
$C = .02\ F$
$E(t) = 100\ V$
$I(0) = 0$
$Q(0) = 0$

IVP for $Q(t)$:

IVP for $I(t)$: