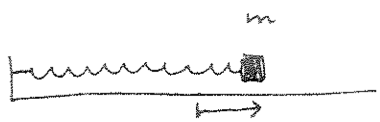
Math 2250-1
Tues 10/29

§5.4 (unforced) mechanical vibrations, $x = x(t)$, solutions to

$$\boxed{m x'' + c x' + k x = 0}$$

← this DE actually arises in several different contexts (including electric circuits), as we shall see... pendulum tomorrow.

<u>model</u>



$x(t)$ displacement from equilibrium.

↑
equilibrium location

Newton's 2^{nd} Law: $m x''(t) = $ net forces

$$= -c x' - k x + F(t)$$

$-cx'$: linear drag, proportional to velocity

$-kx$: force from compression or stretching of spring, relative to equilibrium $x = 0$

$F(t)$: possible external forcing function

(this is really a "linearization" of what is probably a more complication force model, ~~valid~~ but is likely valid for x', x close enough to zero)

in case $F(t) \equiv 0$, this yields

$m x'' + c x' + k x = 0$ for $x(t)$

try $x = e^{rt}$

$$L(x) = \underbrace{(m r^2 + c r + k)}_{p(r).} e^{rt}$$

there are various cases, depending on values of m, c, k.

Case 1 $c = 0$: free undamped motion

$$mx'' + kx = 0$$

$$x'' + \frac{k}{m}x = 0 \qquad\qquad x'' + \omega_0^2 x = 0$$

$$x = e^{rt} \longrightarrow p(r) = r^2 + \frac{k}{m}$$

$$r^2 + \frac{k}{m} = 0$$

$$r^2 = -\frac{k}{m} \Longrightarrow r = \pm i\sqrt{\frac{k}{m}} \longrightarrow \text{soltn } \; x(t) = c_1 \cos\sqrt{\frac{k}{m}}\,t + c_2 \sin\sqrt{\frac{k}{m}}\,t$$

$$\omega_0 := \sqrt{\frac{k}{m}}, \quad c_1 = A, \; c_2 = B$$

$$\boxed{x(t) = A\cos\omega_0 t + B\sin\omega_0 t}$$

trigonometry:

$$A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \alpha)$$

C = amplitude
α = phase angle.

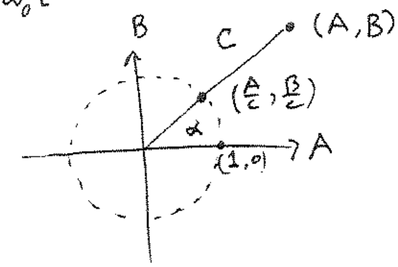
$$C\cos(\omega_0 t - \alpha) = C\left[\cos\omega_0 t\cos(-\alpha) - \sin\omega_0 t\sin(-\alpha)\right]$$
$$= (C\cos\alpha)\cos\omega_0 t + (C\sin\alpha)\sin\omega_0 t$$

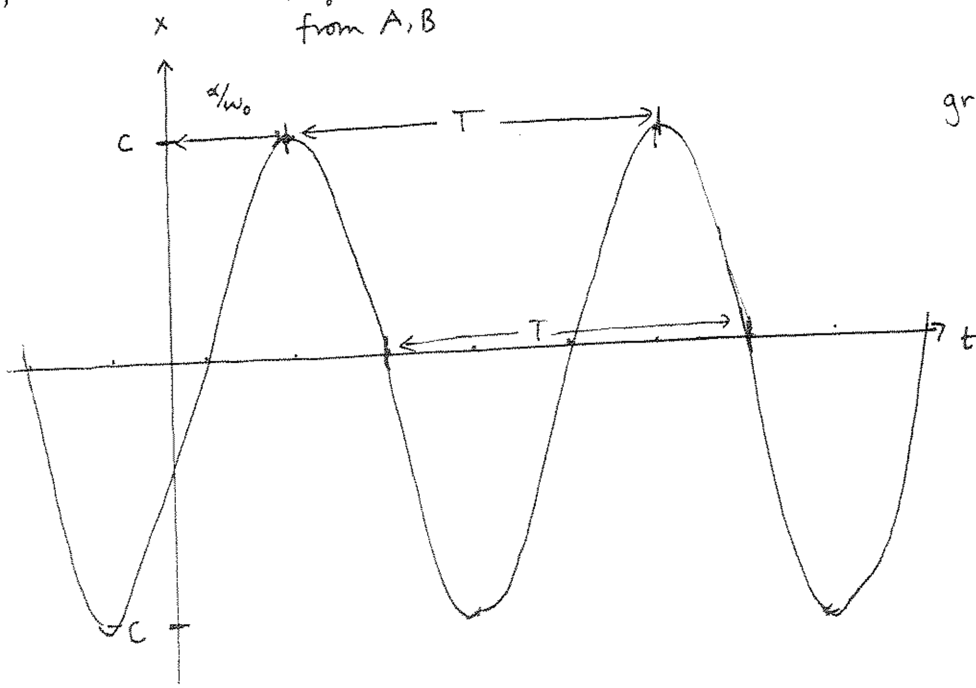
$A = C\cos\alpha$
$B = C\sin\alpha$

↑ to get A, B from C, α

so $C = \sqrt{A^2 + B^2}$
$\cos\alpha = A/C$
$\sin\alpha = B/C$

↑ to get C, α from A, B





graph of $x = C\cos(\omega_0 t - \alpha)$
$= C\cos(\omega_0(t-\delta$

C = amplitude
ω_0 = angular frequenc
(e.g units $\frac{\text{rad}}{\text{s}}$
α = phase angle
$\delta = \frac{\alpha}{\omega_0}$ = time delay
(e.g. units = s

$T = \frac{2\pi}{\omega_0}$ $\quad \frac{\text{seconds}}{\text{cycle}}$

$f = \frac{\omega_0}{2\pi}$ $\quad \frac{\text{cycles}}{\text{second}}$.

Exercise 1 A mass of 2 kg oscillates without damping on a spring with Hooke's constant 18 Newtons/meter. It is initially stretched 1 meter from equilibrium, and released with a velocity of 3/2 m/sec. Show that the displacement x(t) solves

A) $$\begin{cases} x'' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$$

B) Solve the IVP (A). Express the solution in amplitude-phase form, and interpret the (t, x(t)) graph below (Identify amplitude, phase, time delay).

```
> with(plots):
  Digits:=5:
> omega:=3;                #angular frequency
  alpha:=arctan(.5);       #phase
  delta:=arctan(.5)/3;     #time delay
  C:=sqrt(5./4);           #amplitude
  T:=evalf(2*Pi/3);        #period
```

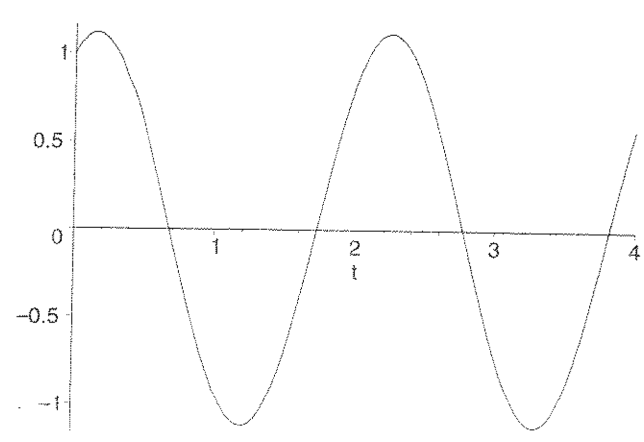
0.46365

0.15455

1.1180

$T := 2.0944$

```
> plot1:=plot(cos(3*t)+.5*sin(3*t),t=0..4,color=black):
  plot2:=plot(C*cos(3*t-alpha),t=0..4,color=black):
  plot3:=plot(C*cos(3*(t-delta)),t=0..4,color=black):
  display({plot1,plot2,plot3});
```



Case 2 : Damping

$$mx'' + cx' + kx = 0$$

$$x'' + 2px' + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\tfrac{k}{m}} \text{ still}; \quad \tfrac{c}{m} = 2p; \quad p = \tfrac{c}{2m}$$

$$p(r) = r^2 + 2pr + \omega_0^2 = 0$$

$$r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2}$$

$$\boxed{r = -p \pm \sqrt{p^2 - \omega_0^2}}$$

Overdamped : $p^2 - \omega_0^2 > 0$ $(c^2 > 4km)$

$$r = r_1 < r_2 < 0$$

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_1 t}\left(c_1 + c_2 e^{(r_2 - r_1)t}\right)$$

Figure 5.4.7 p 331

→ sol's decay exponentially to zero, cross t-axis at most 1 time.

Critically damped $p^2 - \omega_0^2 = 0$ $(c^2 = 4km)$

$r = -p$ double root

$$x(t) = c_1 e^{rt} + c_2 t e^{rt} = e^{rt}(c_1 + c_2 t)$$

→ sol's decay exponentially to zero, cross t-axis at most once

Figure 5.4.8 p 332

Underdamped $p^2 - \omega_0^2 < 0$

set $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$

$$r = -p \pm i\omega_1$$

$$e^{rt} = e^{(-p \pm i\omega)t}$$

$$x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$$

$$= e^{-pt}(C\cos(\omega_1 t - \alpha))$$

Figure 5.4.9 p. 332

① solution oscillates, but oscillations are damped exponentially

② motion is slowed relative to no damping ($\omega_1 < \omega_0$).

pseudo period $T = \frac{2\pi}{\omega_1}$

pseudo freq $f = \frac{\omega_1}{2\pi}$

pseudo ang. freq ω_1

Ce^{-pt} : " time varying amplitude

Exercise 2: Solve and discuss

2a) $\begin{cases} x'' + 6x' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$

2b) $\begin{cases} x'' + 10x' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$

Solve & discuss

```
> deqtn2:=diff(x(t),t,t)+6*diff(x(t),t)+9*x(t)=0:
  IC:=x(0)=1,D(x)(0)=1.5:
  dsolve({deqtn2,IC},x(t)); #critical damping
```

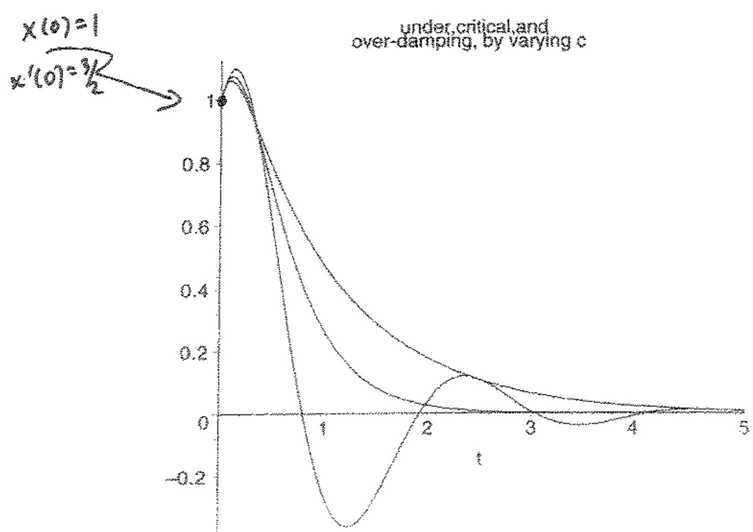
$$x(t) = e^{(-3t)} + \frac{9}{2} e^{(-3t)} t$$

```
> deqtn3:=diff(x(t),t,t)+10*diff(x(t),t)+9*x(t)=0:
  IC:=x(0)=1,D(x)(0)=1.5:
  dsolve({deqtn3,IC},x(t)); #overdamping
```

$$x(t) = \frac{21}{16} e^{(-t)} - \frac{5}{16} e^{(-9t)}$$

```
> plot5:=plot(exp(-3*t)+9/2*exp(-3*t)*t,t=0..5,color=black):
  plot6:=plot(21/16*exp(-t)-5/16*exp(-9*t),t=0..5,color=black):
  display({plot5,plot6,plot2},title=`under,critical,and
  over-damping, by varying c`);
```

2c) $\begin{cases} x'' + 2x' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$



Soltn $x(t) = \frac{5}{4\sqrt{2}} e^{-t} \sin(2\sqrt{2}t) + e^{-t} \cos(2\sqrt{2}t)$