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Math 2250-1
Monday 10/24

§5.3 cont'd : complex roots!
finding the general soltn to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

need n solutions for a basis. (n independent soltns.)

Try $y = e^{rx} \longrightarrow p(r) = r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0$ characteristic polynomial

<u>Case I</u> n distinct roots to $p(r) = 0 \longrightarrow y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$

<u>Case II</u> repeated roots $\longrightarrow$ if $(r - r_1)^k$ is a factor of $p(r)$,
get soltns $e^{r_1 x}, x e^{r_1 x}, \dots x^{k-1} e^{r_1 x}$
amalgamate these to get y_H basis

<u>Case III</u> complex roots:

exponential functions still work — you just need to learn Euler's formula

Recall Taylor-Maclaurin formula from Calculus

$$f(x) \sim f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \dots$$

[notice that the partial sum $P_n(x)$ matches
$f(0), f'(0), f''(0), \dots f^{(n)}(0)$
$= p(0) \quad p'(0) \quad p''(0) \quad p^{(n)}(0)$.]

<u>Exercise 1</u> : Use Taylor-Maclaurin series for recalling

cos x =

sin x =

e^x =

Exercise 2 Use the Taylor series on the previous page, substitute $i\theta$ into the exponential series, to motivate $\boxed{\text{Euler's formula}}$

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta}$$

From Euler's formula, it makes sense to define

$$e^{a+bi} := e^{a} e^{bi} = e^{a}(\cos b + i\sin b) = e^{a}\cos b + i e^{a}\sin b \qquad a, b \in \mathbb{R}$$

So $$e^{(a+bi)x} = e^{ax+ibx} = e^{ax}(\cos bx + i\sin bx) = e^{ax}\cos bx + i e^{ax}\sin bx \qquad \& \; x \in \mathbb{R}$$

$$\boxed{e^{(a+bi)x} := e^{ax}\cos bx + i e^{ax}\sin bx}$$

for a complex function $f(x) + i g(x)$

define $\boxed{\frac{d}{dx}\left(f(x) + i g(x)\right) := f'(x) + i g'(x).}$

Exercise 3 Use the definitions above to check

$$\boxed{\frac{d}{dx}\left(e^{(a+bi)x}\right) = (a+bi)e^{(a+bi)x}}$$

i.e. $\frac{d}{dx} e^{rx} = r e^{rx}$ even if r is complex.

Now return to finding $y(x)$ s.t.

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0.$$

and consider

$$y(x) = f(x) + i\,g(x).$$

$$L(y) = L(f) + i\,L(g) \qquad \text{since } \tfrac{d}{dx} \text{ and } L \text{ are linear.}$$

if r is a complex root of $p(r)$, i.e. $r = a+bi$,

then $\frac{d}{dx} e^{rx} = re^{rx}$ (previous page)

$$\frac{d^2}{dx^2} e^{rx} = r^2 e^{rx}$$

$$\vdots$$

So $L(e^{rx}) = \underbrace{(r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0)}_{=0 \text{ if } r \text{ is a complex root.}} e^{rx}$ just as before

Thus if $p(a+bi) = 0$,

$$L\left(e^{(a+bi)x}\right) = 0 = 0 + 0i$$

$$L\left(e^{ax}\cos bx + i e^{ax}\sin bx\right) = 0 + 0i$$

$$L(e^{ax}\cos bx) + i\,L(e^{ax}\sin bx) = 0 + 0i$$

<u>upshot</u> if $a+bi$ is a complex root of the characteristic polynomial,

$$e^{ax}\cos bx,\ e^{ax}\sin bx$$

are two linearly independent <u>real</u> function solutions

($a-bi$ will also be a root, gives an equivalent basis for this 2-d subspace.

<u>multiple roots</u>

if $(r-(a+bi))^k$ is a factor of $p(r)$ then

$$e^{ax}\cos bx,\ xe^{ax}\cos bx, \dots\ x^{k-1}e^{ax}\cos bx$$
$$e^{ax}\sin bx,\ xe^{ax}\sin bx, \dots\ x^{k-1}e^{ax}\sin bx$$

are $2k$ independent real sol'ns.

($p(r)$ will also have the factor $(r-(a-bi))^k$, which gives an equivalent basis for this $2k$-dimensional subspace.)

<u>Exercise 4</u> Recover $\{\cos 2x, \sin 2x\}$ as a basis for the solution space to

$$y'' + 4y = 0$$

using the characteristic poly and Euler's formula

<u>Exercise 5</u> Find the general solution to

$$y'' + 2y' + 5y = 0.$$

<u>Exercise 6</u> Find the general soltn to

$$y''' - y'' - y' - 15y = 0$$

Hint: $r^3 - r^2 - r - 15 = (r-3)(r^2 + 2r + 5)$

<u>Exercise 7</u> What if L is 4^{th} order, with characteristic polynomial $p(r) = (r^2 + 25)^2$?
Find a basis for the sol'n space to $L(y) = 0$
What is L?