Math 2250-1
Wed 10/19

↳ 5.2 n^{th} order linear DE's.

- Finish exercise 6 on Tuesday's notes; (page 3)
  this example illustrates how different choices of basis can be useful for different IVP's, for homogeneous linear DE's.

- See how n^{th} order homogeneous linear DE's has theory analogous to 2^{nd} order.

this is page 4 of Tuesday's notes, and below.

Notice that if we're trying to solve the IVP for $y(x)$:

$$\begin{cases} y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0 \quad \leftarrow \text{homogeneous} \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{cases}$$

with a general soltn

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

then

$$\begin{aligned} b_0 &= c_1 y_1(x_0) + c_2 y_2(x_0) + \ldots + c_n y_n(x_0) \\ b_1 &= c_1 y_1'(x_0) + c_2 y_2'(x_0) + \ldots + c_n y_n'(x_0) \\ &\vdots \\ b_{n-1} &= c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \ldots + c_n y_n^{(n-1)}(x_0) \end{aligned}$$

which is a matrix eqtn

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & & y_n'(x_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

↑

Wronskian matrix for $y_1, y_2 .. y_n$, at $x = x_0$
its determinant is called W, the Wronskian. If $W \neq 0$, $y_1(x), y_2(x), \ldots y_n(x)$ are a bas

Neat shortcut:
If $W \neq 0$ @ x_0
then you can solve all IVP's there with unique linear combo coeffs
$\Rightarrow$ basis $y_1, y_2, .. y_n$
$\Rightarrow W \neq 0$
at <u>all</u> $x \in I$

<u>Superposition</u>, and the general soltns for <u>nonhomogeneous</u> linear DE's.

We've checked that the linear operator

$$L(y) := y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x) y$$

satisfies the properties

(i) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(ii) $L(c y_1) = c L(y_1)$, $c \in \mathbb{R}$.

- These are the properties that let us deduce that the solution space to $L(y) = 0$ is a subspace. (The existence-uniqueness thm for initial value problems led to the conclusion that this subspace is n-dimensional.).

- What about the general solution (and then solving IVP's) for the non-homogeneous n^{th} order linear DE

$$L(y) = f$$

(where f is not the zero function).

<u>Theorem</u> Let L be any operator satisfying (i) and (ii) above.

Let y_H be the general solution to the homogeneous equation

$$L(y) = 0$$

(so, for n^{th} order linear DE's, $y_H(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$)

Let $y_P(x)$ be any single ("particular") solution to

$$L(y_P) = f$$

Then the general solution to

$$L(y) = f$$

is $y = y_P + y_H$

<u>Check Theorem:</u>

- $L(y_P + y_H) = L(y_P) + L(y_H) = f + 0 = f$, so all such $y(x)$ do solve the DE

- If y_Q is any other solution to $L(y) = f$, then $L(y_Q - y_P) = L(y_Q) - L(y_P) = f - f = 0$

So $y_Q - y_P$ is a solution to $L(y) = 0$

So $y_Q = y_P$ plus a soltn to $L(y) = 0$ ∎

Exercise 8.

We will solve the non-homogeneous DE IVP below in steps:

$$\begin{cases} y'' + 4y = 2x - 8 \\ y(0) = 3 \\ y'(0) = 6 \end{cases}$$

text writes $y_c(x)$ for complementary soltn, rather than $y_H(x)$ for homogeneous soltn

step 1 What is the general sol'n $y_H(x)$ to

$$y'' + 4y = 0$$

(we did this!)

step 2 Find a particular soltn of the form $y_p(x) = Ax + B$ (why did such a guess make sense).

step 3 Write down the general soltn to the non homogeneous DE

$$y'' + 4y = 2x - 8$$

step 4 Solve the IVP.