Math 2250-1
Mon. 10/17

§5.1 is about 2^{nd} order linear DE's.
This ties into vector space theory 4.1-4.4 as follows:

Our vectorspace now is not $\mathbb{R}^n$ or its subspaces, rather it is
$$V = C(\mathbb{R}) := \{ f: \mathbb{R} \to \mathbb{R} \text{ s.t. } f \text{ is a } \underline{\text{continuous}} \text{ function} \}$$
and its subspaces

do you remember what this means?

Check that the vector space axioms for linear combinations are satisfied: $(f+g)(x) := f(x) + g(x)$
$(cf)(x) := c \cdot f(x)$

α) if $f, g \in V$ then $f+g \in V$
β) if $f \in V$, $c \in \mathbb{R}$ then $cf \in V$

(a) $f+g = g+f$
(b) $f + (g+h) = (f+g)+h$
(c) $f + 0 = f$ ⟵ the zero function is 0 for all x
(d) $\exists\, -f$ s.t. $f + (-f) = 0$
(e) $c(f+g) = cf + cg$ $\quad c \in \mathbb{R}$
(f) $(c+d)f = cf + df$ $\quad c, d \in \mathbb{R}$
(g) $c(df) = (cd)f$
(h) $1 \cdot f = f$, $(-1) \cdot f = -f$, $0f = 0$

Thus all of the concepts we talked about for $\mathbb{R}^n$ and its subspaces make sense in this context too

- span$\{f_1, f_2, \ldots f_n\}$
- linear independence/dependence
- subspace
- basis.

<u>Def</u> A <u>second order</u> <u>linear</u> differential equation is an equation

$$A(x)\,y'' + B(x)y' + C(x)y = F(x)$$

We search for solutions $y = y(x)$ on some interval $a \le x \le b$, or $[a,\infty)$, $(-\infty,a]$, $(-\infty,\infty)$.
In this chapter we assume $A(x) \neq 0$ on the domain interval, so we can rewrite the DE after dividing by $A(x)$, as

$$y'' + p(x)y' + q(x)y = f(x).$$

One reason this DE is called <u>linear</u> is that the operator

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$
(2) $L(cy) = cL(y)$, $c \in \mathbb{R}$.

<u>Exercise 1</u>a): Check the linearity properties for L, (1), (2).
1b) Use these properties to show that the solutions to the homogeneous DE

$$y'' + p(x)y' + q(x)y = 0$$

form a <u>subspace</u>. (Hint: it's the same mathematical argument that shows the sol'n space to a homogeneous matrix eqtn $A\vec{x} = \vec{0}$ is a subspace.)

Theorem 1

Existence - Uniqueness Theorem ($\exists!$)

for the 2^{nd} order DE

$$y'' + p(x)y' + q(x)y = f(x)$$

if $p(x), q(x), f(x)$ are continuous on the interval I, and $a \in I$, then there is a unique soltn $y(x)$ to

$$\text{IVP} \begin{cases} y''(x) + p(x)y' + q(x) = f(x) \\ y(a) = b_0 \\ y'(a) = b_1 \end{cases}$$

and $y(x)$ exists $\forall x \in I$.

Exercise 2 Verify the $\exists!$ theorem for $I = (-\infty, \infty)$ and

$$\text{IVP} \begin{cases} y'' + 2y' = 0 \\ y(0) = b_0 \\ y'(0) = b_1 \end{cases}$$

hint: this is a 1^{st} order DE for $v = y'$

unlike for the 1st order DE

$$y' + p(x)y = f(x)$$

there is not a clever integrating factor formula for the general soln of the 2nd order linear DE

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to algorithms based on clever guessing.

It will help to know

Theorem 2 The solution space to the second order homogeneous DE

$$L(y) := y'' + p(x)y' + q(x)y = 0$$

on the interval I is 2-dimensional

Exercise 3 (see #2 from 9/30 notes). Consider the DE

$$y'' - 2y' - 3y = 0.$$

a) find two exponential functions $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$ that solve this DE

b) show that every IVP

$$y(0) = b_0$$
$$y'(0) = b_1$$

can be solved with a unique linear combo $y(x) = c_1 y_1(x) + c_2 y_2(x)$.
Deduce that $\{y_1(x), y_2(x)\}$ are a basis for the soln space

Theorem 2 actually follows from Theorem 1.
Here's why. Also note the new words that we'll use in this chapter.
(this is really just the generalization of what we did on page 4.)

Why the solution space to

$$* \quad y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional:

pick $x_0 \in I$.
find soltns to IVP, $y_1(x)$ and $y_2(x)$, so that the matrix

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

← this is called the Wronskian matrix $W(y_1, y_2)$, at x_0
its determinant is called the Wronskian

is invertible.
(i.e. $\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}$ are a basis for $\mathbb{R}^2$)

you might be able to find $y_1(x), y_2(x)$ by good guessing, as in previous example, but the $\exists!$ Theorem 1 guarantees they exist.

under these conditions, $\{y_1(x), y_2(x)\}$ are a basis for the solution space:

- span: the condition that the Wronskiam matrix is invertible means we can solve every IVP with a linear combo of y_1 & y_2:

  for $y(x_0) = b_0$
  $y'(x_0) = b_1$

  try $y(x) = c_1 y_1 + c_2 y_2$
  $(\Rightarrow \; y'(x) = c_1 y_1' + c_2 y_2'$

  $$@\, x_0: \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

  $$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

  since Theorem 1 uniqueness says each IVP has exactly 1 soltn, so if $y(x)$ is any soltn to $*$, with $y(x_0) = b_0$, $y'(x_0) = b_1$, it has to be $c_1 y_1 + c_2 y_2$ as above.

- independent: $c_1 y_1 + c_2 y_2 \equiv 0$
  $\Rightarrow c_1 y_1' + c_2 y_2' \equiv 0$ ↖ @ $x = x_0$, this is

  $$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$