Numerical approximations to DE solutions

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

approximate soltns on interval $[x_0, x_n]$ with n subintervals of length $h = \frac{x_n - x_0}{n}$.

Euler

$k = f(x_j, y_j)$

$x_{j+1} = x_j + h$

$y_{j+1} = y_j + hk$

Improved Euler

$k_1 = f(x_j, y_j)$

$k_2 = f(x_j + h, y_j + hk_1)$

$k = \frac{1}{2}(k_1 + k_2)$

$x_{j+1} = x_j + h$

$y_{j+1} = y_j + hk$

Runge Kutta

$k_1 = f(x_j, y_j)$

$k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$

$k_3 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_2)$

$k_4 = f(x_j + h, y_j + hk_3)$

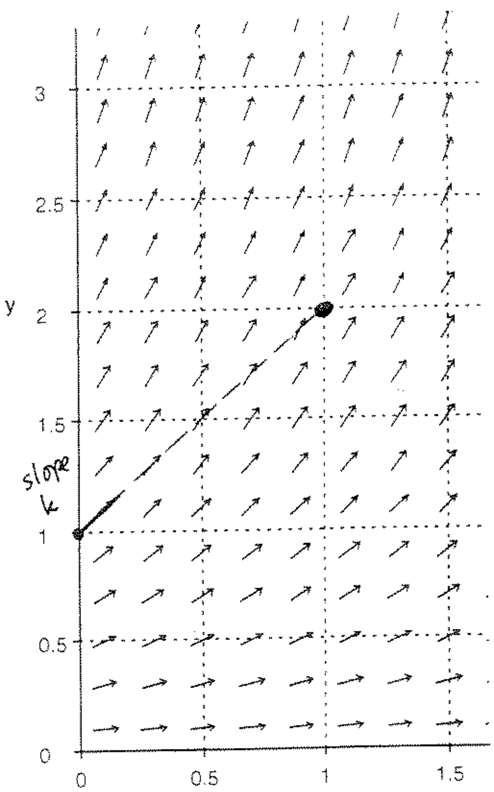
$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

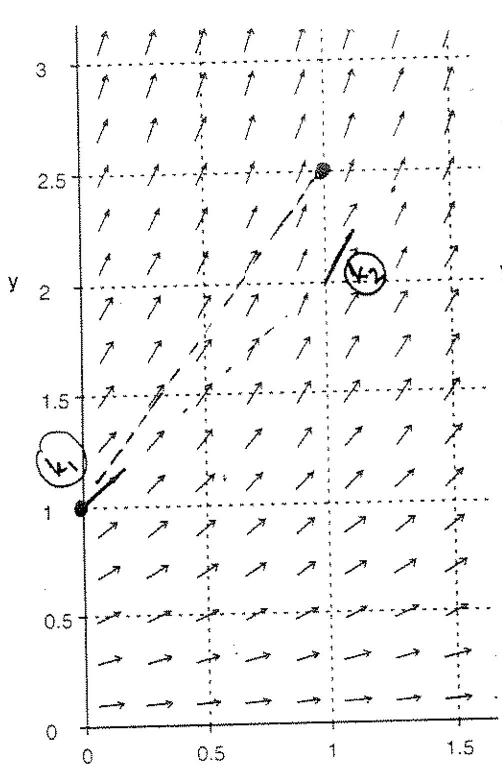
$x_{j+1} = x_j + h$

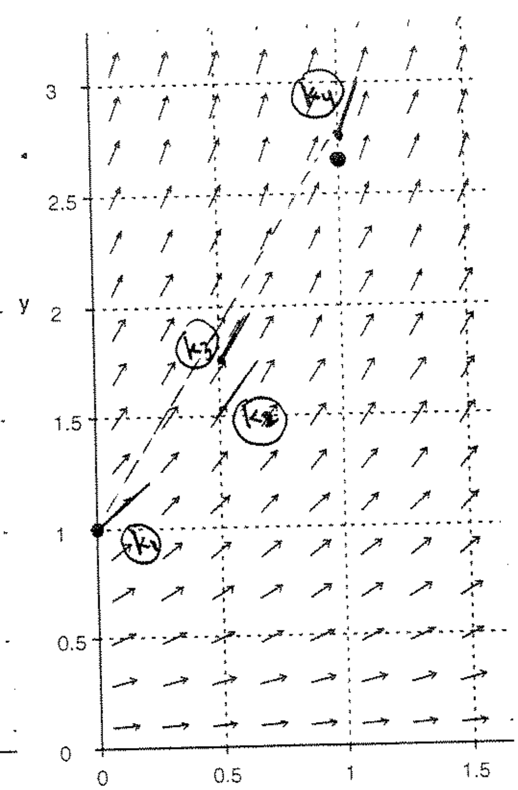
$y_{j+1} = y_j + hk$

illustrated for $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$

$n = 1$, interval $[0, 1]$.







$x_0 = 0$

$y_0 = 1$

$k = 1$

$x_1 = 1$

$y_1 = 1 + 1 \cdot 1 = \boxed{2}$

est. for e

Euler

$k_1 = 1$

$k_2 = 2$

$k = \frac{1}{2}(k_1 + k_2) = 1.5$

$y_1 = 1 + (1.5)$

$= \boxed{2.5}$

Improved Euler

$k_1 = 1$

$k_2 = 1.5$

$k_3 = 1.75$

$k_4 = 2.75$

$k = \frac{1}{6}(1 + 2k_1 + 2k_2 + k_3)$

$\cong 1.708$

$\boxed{y_1 \cong 2.708}$

Runge Kutta!