Math 2250-1
Monday 11/7

Decide on time for going over old exam - when on Tuesday?

10.1 - 10.3 cont'd.

Continue filling in the Laplace transform table and doing examples

exercise 1

0

$\mathcal{L}\{t\}(s) = \frac{1}{s^2}$

$\mathcal{L}\{t^2\}(s) = \frac{2}{s^3}$

$\mathcal{L}\{t^n\}(s) = \frac{n}{s}\mathcal{L}\{t^{n-1}\}(s)$

$= n!/s^{n+1}$

exercise 2

1b $\mathcal{L}\{\cosh(kt)\}(s)$

$\mathcal{L}\{\sinh(kt)\}(s)$

	$f(t)$	$F(s) := \int_0^\infty e^{-st} f(t)\,dt$ $\quad s > M$	checked
	$c_1 f(t) + c_2 g(t)$	$c_1 F(s) + c_2 G(s)$	✓
0	1	$1/s$	✓
	t	$1/s^2$	
	t^n	$n!/s^{n+1}$ $\quad n \in \mathbb{N}$	
	$e^{\alpha t}$		✓
1	$\cos kt$		✓
	$\sin kt$		✓
	$\cosh kt$		
	$\sinh kt$		
	$f'(t)$	$sF(s) - f(0)$	✓
	$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$	✓
	$f'''(t)$	$s^3 F(s) - s^2 f(0) - sf'(0) - f(0)$	
2 analogous!	etc.		
	$\int_0^t f(\tau)\,d\tau$	$F(s)/s$	
	$t f(t)$	$-F'(s)$	
	$t^2 f(t)$	$F''(s)$	
	$t^n f(t)$	$(-1)^n F^{(n)}(s)$ $\quad n \in \mathbb{N}$	
	$f(t)/t$	$\int_s^\infty F(\sigma)\,d\sigma$	
	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2 + k^2}$	✓
	$e^{at}\sin kt$	$\frac{k}{(s-a)^2 + k^2}$	✓
3	$e^{at} f(t)$	$F(s-a)$	✓
	$u(t-a)$	e^{-as}/s	
	$u(t-a) f(t-a)$	$e^{-as} F(s)$	
applications of 2b; good for resonance problems	$t\cos kt$	$\frac{s^2 - k^2}{(s^2+k^2)^2}$	
	$\frac{1}{2k} t \sin kt$	$\frac{s}{(s^2+k^2)^2}$	
	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$	
	$t e^{at}$	$\frac{1}{(s-a)^2}$	
	etc.		

MORE to appear!

exercise 3

finish 2a : $\mathcal{L}\{f'''(t)\}(s) = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{F(s)}{s}$$

exercise 4 Use $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{F(s)}{s}$

to compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t)$.

Compare to using partial fractions

computation to check:

$$\mathcal{L}\{t f(t)\}(s) = -F'(s)$$

check

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\int_0^\infty e^{-(s+\Delta s)t} f(t)\,dt - \int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= \lim_{\Delta s \to 0} \frac{1}{\Delta s}\int_0^\infty e^{-(s+\Delta s)t} f(t) - e^{-st} f(t)\;dt$$

$$= \lim_{\Delta s \to 0} \int_0^\infty \left(\frac{e^{-(s+\Delta s)t} - e^{-st}}{\Delta s}\right) f(t)\,dt$$

$$\overset{*}{=} \int_0^\infty \underbrace{\lim_{\Delta s \to 0}\left(\frac{e^{-(s+\Delta s)t} - e^{-st}}{\Delta s}\right)}_{\frac{d}{ds}\left(e^{-st}\right) \,=\, -t e^{-st}} f(t)\,dt$$

* (think of the integral as a sum, and the idea is that the limit of a sum is the sum of the limits)

$$= -\int_0^\infty e^{-st}\, t f(t)\,dt$$

i.e. $-F'(s) = \mathcal{L}\{t f(t)\}(s)$. $\blacksquare$

$$\Rightarrow \mathcal{L}\{t^2 f(t)\}(s) = \mathcal{L}\{t\cdot(t f(t))\}(s)$$

$$= -\left(-F'(s)\right)'$$

$$= F''(s) \quad \text{etc.}$$

Exercise 5 Use $\mathcal{L}\{t f(t)\}(s) = -F'(s)$

to compute $\mathcal{L}\{t\cosh kt\}(s) = \dfrac{s^2 - k^2}{(s^2 + k^2)^2}$

$\mathcal{L}\left\{\frac{1}{2k}\, t \sin kt\right\}(s) = \dfrac{s}{(s^2 + k^2)^2}$

computation to check

$$\frac{1}{(s^2+k^2)^2} = \frac{1}{2k^2}\left[\frac{s^2+k^2}{(s^2+k^2)^2} - \frac{s^2-k^2}{(s^2+k^2)^2}\right]$$

so $$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}(t) = \frac{1}{2k^2}\left[\frac{1}{k}\sin kt - t\cos kt\right]$$

$f(t)$	$F(s)$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$\frac{1}{2k}t\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$\frac{1}{2k^3}[\sin kt - kt\cos kt]$	$\frac{1}{(s^2+k^2)^2}$

<u>Exercise 6</u> Resonance revisited!

Solve the IVP for $x(t)$

$$\begin{cases} x''(t) + \omega_0^2 x(t) = F_0 \sin \omega t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$\omega = \omega_0$
(resonance)

$\omega \neq \omega_0$
(no resonance)