Math 2250-1
Wed 11/30

Chapter 9 introduction : non-linear systems of DE's.

e.g.

$$(1)\quad \begin{cases} \frac{dx}{dt} = F(x,y,t) \\ \frac{dy}{dt} = G(x,y,t) \end{cases}$$

system of 2 1^{st} order DE's.

example (9.3):

$x(t)$ = prey population (fish, rabbits, etc.)

$y(t)$ = predator population (sharks, foxes, etc.)

$$\begin{cases} \frac{dx}{dt} = ax - pxy \\ \frac{dy}{dt} = -by + qxy \end{cases}$$

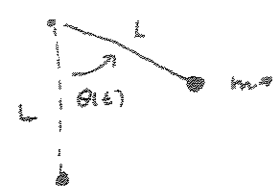
($-cx^2$ if you want logistic prey)

explain model assumptions:

example (9.4):

$x(t)$ = pendulum angle $\theta(t)$

$y(t) = x'(t)$ = angular velocity



we've derived

$$x'' + \frac{g}{L}\sin x = 0, \text{ so}$$

$$\begin{cases} x' = y \\ y' = -\frac{g}{L}\sin x \end{cases}$$

Def: If the only dependence of F & G on t in (1) is through $x(t)$ & $y(t)$ (as in the 2 examples above), the system is called **autonomous**, i.e.

$$(2)\quad \begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$$

In this case we call the x-y plane the **phase plane**, and the solution curves $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ are called **trajectories**. They follow the **tangent vector field** $\begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$.

constant sol'ns to (2) are called equilibrium solutions
They are exactly the sol'ns to the (nonlinear) system

$$(3) \quad \begin{cases} 0 = F(x,y) \\ 0 = G(x,y) \end{cases}$$

example competing species, say $x(t)$ = rabbit population
$y(t)$ = squirrel population

perhaps

$$\frac{dx}{dt} = 14x - 2x^2 - xy$$

$$\frac{dy}{dt} = \underbrace{16y - 2y^2}_{\text{logistic}} - \underbrace{xy}_{\text{competition.}}$$

Find the equilibrium sol'ns:

ans (0,0)
(0,8)
(7,0)
(4,6)

It will be important to know whether equilibrium sol'ns are stable or unstable:

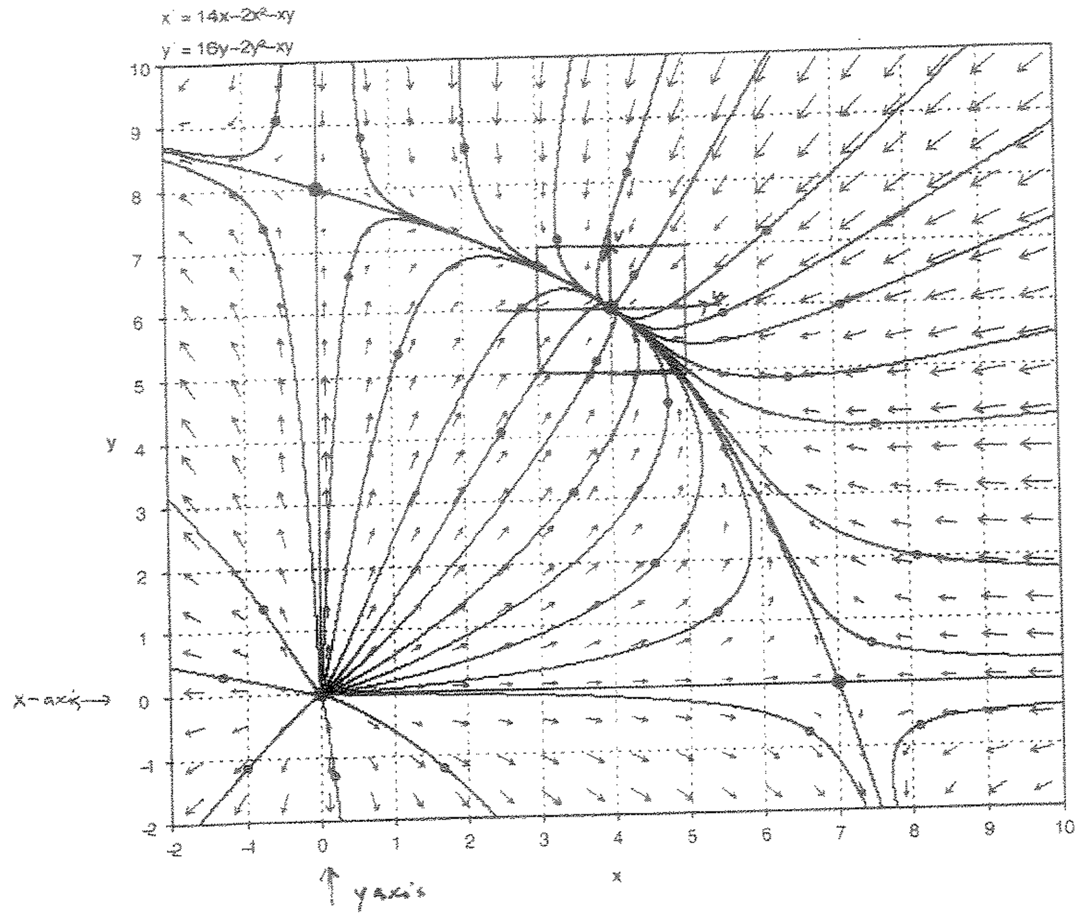
Def $\begin{bmatrix} x_* \\ y_* \end{bmatrix} = \vec{x}^*$ is a stable equilibrium for (2) if it is a const (equil) solution, i.e. satisfies (3),

and if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.
whenever $\|\vec{x}^* - \vec{x}_0\| < \delta$ $\quad \left(\|\vec{x}^* - \vec{x}_0\| = \sqrt{(x_0 - x_*)^2 + (y_0 - y_*)^2} \right)$
then the sol'n to (2) with $\vec{x}(0) = \vec{x}_0$
satisfies $\|\vec{x}(t) - \vec{x}^*\| < \varepsilon \quad \forall t > 0$.

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is unstable equilibrium if it is an equilibrium sol'n that is not stable.

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is asymptotically stable iff it's a stable equilibrium, and $\exists \delta > 0$
s.t. $\|\vec{x}^* - \vec{x}_0\| < \delta \Rightarrow$ sol'n to IVP with $\vec{x}(0) = \vec{x}_0$
satisfies $\lim_{t \to \infty} \vec{x}(t) = \vec{x}^*$

Here's the phase portrait (made with pplane) for our competition model:



What happens to rabbit-squirrel populations??

- notice that if we restrict this picture to either x-axis or y-axis (i.e. one pop ≡ 0) we get the Chapter 2 1-dim'l phase portraits for logistic DE's.

- discuss apparent stability of the 4 equilibria.

- discuss apparent long term population behavior, assuming both $x_0, y_0 > 0$.

<u>linearization</u>: Near an equilibrium point, the non-linear system can be effectively approximated with a linear one:

<u>Example</u>: linearize rabbit-squirrel model near $\begin{bmatrix}4\\6\end{bmatrix}$:

let $x = 4 + u$ $\quad u = u(t)$ small in abs. val.
$\quad y = 6 + v$ $\quad v = v(t)$ small " "

then
$$\frac{du}{dt} = \frac{dx}{dt} = 14x - 2x^2 - xy = 14(4+u) - 2(4+u)^2 - (4+u)(6+v)$$
$$= \cancel{56} + 14u \cancel{-32} - 16u - 2u^2 \cancel{-24} - 6u - 4v - uv$$
$$= -8u - 4v - 2u^2 - uv$$

$$\frac{dv}{dt} = \frac{dy}{dt} = 16y - 2y^2 - xy = 16(6+v) - 2(6+v)^2 - (4+u)(6+v)$$
$$= \underbrace{96 - 72 - 24}_{0} + u(-6) + v(16 - 24 - 4) + v^2(-2) + uv(-1)$$
$$= -6u - 12v - 2v^2 - uv$$

$$\begin{bmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{bmatrix} = \begin{bmatrix}-8 & -4\\ -6 & -12\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} + \begin{bmatrix}-2u^2 - uv\\ -2v^2 - uv\end{bmatrix}$$

linear piece

error: if $\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\| < \delta$, then $\|\text{error}\| \le 3\sqrt{2}\,\delta^2$
$\quad$ is small $\quad$ is tiny.

in the linearization, discard the error term!

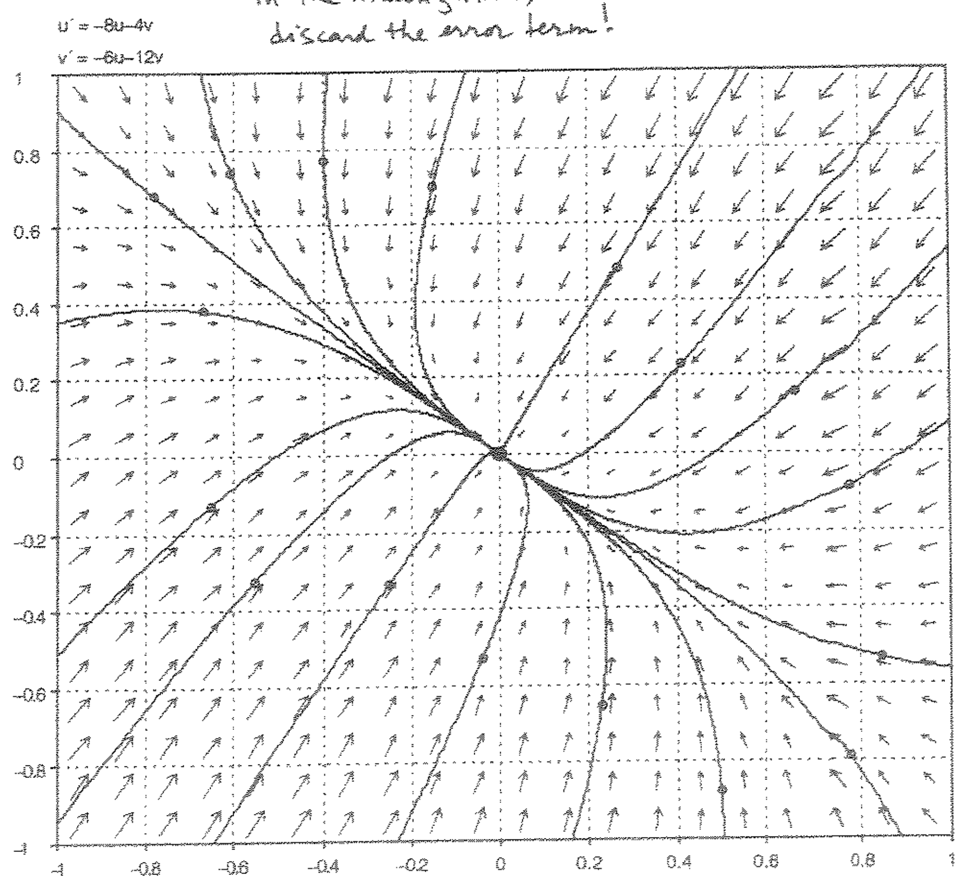
Matrix A has eigendata:

$\lambda = -4.7$, $\lambda = -15.3$

$\vec{v}_1 = \begin{bmatrix}.77\\-.64\end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix}.49\\.89\end{bmatrix}$

$$\begin{bmatrix}u(t)\\v(t)\end{bmatrix} \approx c_1 e^{-4.7t}\vec{v}_1 + c_2 e^{-15.3t}\vec{v}_2$$

notice how the translation change of variables and linearization yields a picture which essentially just magnifies the little box on page 3!



Shortcut to linearization: (works for systems of n DE's; illustrated for $n=2$)

let
$$(1)\quad \begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

$$F(x_*,y_*) = F(P) = 0$$
$$G(x_*,y_*) = G(P) = 0$$

write $\quad x(t) = x_* + u(t)$
$\qquad\quad\; y(t) = y_* + v(t)$

we are interested in what happens for $\|(u,v)\|$ small.

error; $\frac{\varepsilon}{\|(u,v)\|} \to 0$ as $(u,v)\to(0,0)$

$$x' = F(x_*+u,\, y_*+v) = \cancelto{0}{F(x_*,y_*)} + F_x(x_*,y_*)u + F_y(x_*,y_*)v + \varepsilon_1(u,v)$$
$$y' = G(x_*+u,\, y_*+v) = \cancelto{0}{G(x_*,y_*)} + G_x(-)u + G_y(-)v + \varepsilon_2(u,v)$$

Math 2210 affine approx.

$$u' = x' = F_x u + F_y v + \varepsilon_1(u,v)$$
$$v' = y' = G_x u + G_y v + \varepsilon_2(u,v)$$

where the partial derivs of F & G are evaluated at the equil. pt.

$$(2)\qquad \begin{bmatrix} u' \\ v' \end{bmatrix} \cong \underbrace{\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}_{\text{"A"}} \begin{bmatrix} u \\ v \end{bmatrix}$$

(Partial derivs in A are evaluated at (x^*,y^*))

this is the linearization of (1), at (x^*,y^*). (We use the same letters u, v as we did for the non-linear problem, even though the sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the linearized problem only approximates the translated sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the non-linear problem)

the eigenvector data of A determines stability for the nonlinear system (1), in the non borderline cases.

the matrix A is called the <u>Jacobian matrix</u> for $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$, at $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$

<u>Exercise</u>

Check your page 4 work with the Jacobian shortcut!