Math 2250-1

Tues 11/29

§7.4 cont'd : undamped spring systems

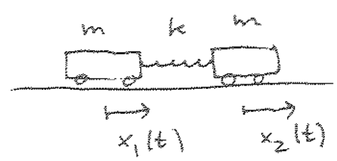
- $\lambda = 0$ in "train" problems
- forced oscillations
- transverse oscillations and your earthquake project.

If coupled masses & springs aren't tethered to any walls, then in the unforced undamped system

$$\vec{x}'' = A\vec{x}$$

the eigenvalue $\lambda = 0$ will occur. In this case $\omega = \sqrt{-\lambda}$ doesn't make sense for sinusoidal motion, but other natural solutions occur.

<u>Exercise 1</u> : Consider the "train"



(In 7.4.16, 18 you consider $m_1 \neq m_2$)

a) Show

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{k}{m} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

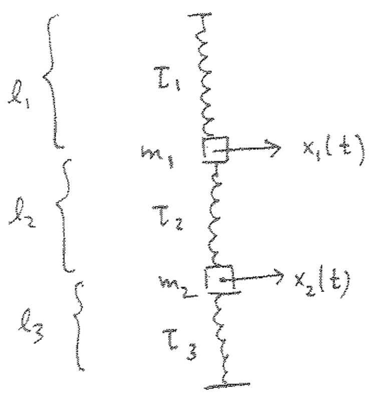
b) Find the general solution. Verify and use the fact that if $\vec{v}$ is an eigenvector of A with eigenvalue $\lambda = 0$, then

$$\vec{x}(t) = (c_1 + c_2 t)\vec{v} \qquad \text{solves} \qquad \vec{x}''(t) = A\vec{x}$$

(because both sides $= 0$!)

- Forced oscillations and practical resonance in mass-spring systems.
  - This is pages 5-7 Monday notes

- Transverse oscillations! (i.e. directions $\perp$ to the mass-spring configuration)



T_1, T_2, T_3 are the tensions (pulling forces) of the stretched springs

By linearization, a good model would be

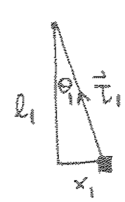
$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1+K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = K_2(x_1 - x_2) - K_3 x_2 = K_2 x_1 - (K_2+K_3)x_2$$

where K_1, K_2, K_3 are positive constants as before

→ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2^{nd} order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & fundamental modes.

[force picture, e.g.



horiz force from top spring on mass 1

$$= -T_1 \sin\theta_1 = -T_1 \frac{x_1}{\sqrt{l_1^2 + x_1^2}} \approx -T_1 \frac{x_1}{l_1} = -\frac{T_1}{l_1} x_1$$

So $K_1 = \frac{T_1}{l_1}$

similarly, $K_2 = \frac{T_2}{l_2}$, $K_3 = \frac{T_3}{l_3}$

for our physics demo springs, equilibrium length ≈ 0, very Hookesian so $T \cong k l$; $\frac{T}{l} \approx k$, so actually almost recover same fundamental modes !!]

(3)

26. Suppose that $k_1 = k_2 = k$ and $L_1 = L_2 = \frac{1}{2}L$ in Fig. 7.4.14 (the symmetric situation). Then show that every free oscillation is a combination of a vertical oscillation with frequency

$$\omega_1 = \sqrt{2k/m}$$

and an angular oscillation with frequency

$$\omega_2 = \sqrt{kL^2/(2I)}.$$

In Problems 27 through 29, the system of Fig. 7.4.14 is taken as a model for an undamped car with the given parameters in fps units. **(a)** *Find the two natural frequencies of oscillation (in hertz).* **(b)** *Assume that this car is driven along a sinusoidal washboard surface with a wavelength of 40 ft. Find the two critical speeds.*

27. $m = 100$, $I = 800$, $L_1 = L_2 = 5$, $k_1 = k_2 = 2000$

28. $m = 100$, $I = 1000$, $L_1 = 6$, $L_2 = 4$, $k_1 = k_2 = 2000$

29. $m = 100$, $I = 800$, $L_1 = L_2 = 5$, $k_1 = 1000$, $k_2 = 2000$

7.4 Application Earthquake-Induced Vibrations of Multistory Buildings

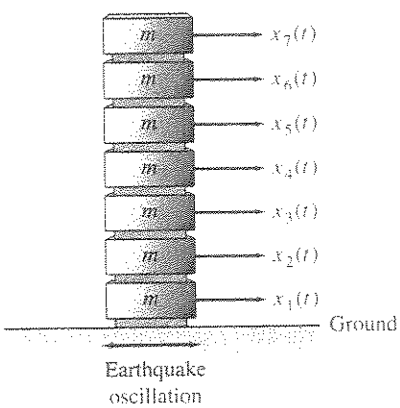


FIGURE 7.4.15. The seven-story building.

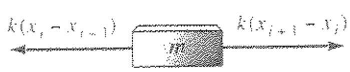
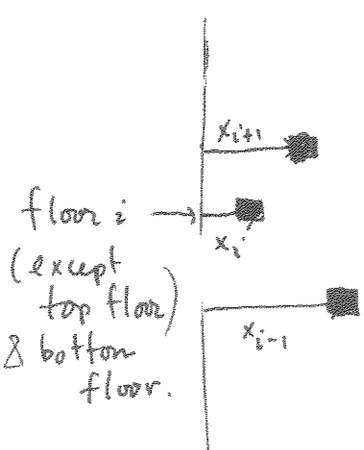


FIGURE 7.4.16. Forces on the ith floor.



In this application you are to investigate the response to transverse earthquake ground oscillations of the seven-story building illustrated in Fig. 7.4.15. Suppose that each of the seven (above-ground) floors weighs 16 tons, so the mass of each is $m = 1000$ (slugs). Also assume a horizontal restoring force of $k = 5$ (tons per foot) between adjacent floors. That is, the internal forces in response to horizontal displacements of the individual floors are those shown in Fig. 7.4.16. It follows that the free transverse oscillations indicated in Fig. 7.4.15 satisfy the equation $\mathbf{Mx}'' = \mathbf{Kx}$ with $n = 7$, $m_i = 1000$ (for each i), and $k_i = 10000$ (lb/ft) for $1 \leqq i \leqq 7$. The system then reduces to the form $\mathbf{x}'' = \mathbf{Ax}$ with

$$\begin{bmatrix} x_1'' \\ x_2'' \\ \vdots \\ \\ x_7'' \end{bmatrix} \cancel{\mathbf{A} =} \begin{bmatrix} \overset{-2k/m}{-20} & \overset{k/m}{10} & 0 & 0 & 0 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & -20 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & -20 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & -20 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 & -20 & 10 \\ 0 & 0 & 0 & 0 & 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ x_7 \end{bmatrix} \leftarrow \text{top floor} \tag{1}$$

Once the matrix $\mathbf{A}$ has been entered, the TI-86 command **eigVl A** takes about 15 seconds to calculate the seven eigenvalues shown in the λ-column of the table in Fig. 7.4.17. Alternatively, you can use the *Maple* command **eigenvals(A)**, the MATLAB command **eig(A)**, or the *Mathematica* command **Eigenvalues[A]**.

i	**Eigenvalue** λ	**Frequency** $\omega = \sqrt{-\lambda}$	**Period** $P = \frac{2\pi}{\omega}$ **(sec)**
1	−38.2709	6.1863	1.0157
2	−33.3826	5.7778	1.0875
3	−26.1803	5.1167	1.2280
4	−17.9094	4.2320	1.4847
5	−10.0000	3.1623	1.9869
6	−3.8197	1.9544	3.2149
7	−0.4370	0.6611	9.5042

FIGURE 7.4.17. Frequencies and periods of natural oscillations of the seven-story building.

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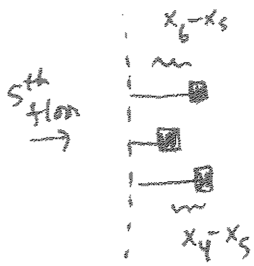

Then calculate the entries in the remaining columns of the table showing the natural frequencies and periods of oscillation of the seven-story building. Note that a typical earthquake producing ground oscillations with a period of 2 seconds is uncomfortably close to the fifth natural frequency (with period 1.9869 seconds) of the building.

A horizontal earthquake oscillation $E \cos \omega t$ of the ground, with amplitude E and acceleration $a = -E\omega^2 \cos \omega t$, produces an opposite inertial force $F = ma = mE\omega^2 \cos \omega t$ on each floor of the building. The resulting nonhomogeneous system is

e.g. 5th floor

$$x_5'' = \frac{k}{m}(x_6 - x_5) + \frac{k}{m}(x_5 - x_4) + E\omega^2 \cos\omega t \cdot 1$$

$$\mathbf{x}'' = \mathbf{A}\mathbf{x} + (E\omega^2 \cos \omega t)\mathbf{b}, \qquad (2)$$

try $\vec{x}_p = \cos\omega t\, \vec{c}$. find $\vec{c} = \vec{c}(\omega)$.

ground acceleration is $-E\omega^2 \cos\omega t$, so to a ground observer, floor has opposite acceleration

where $\mathbf{b} = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,]^T$ and $\mathbf{A}$ is the matrix of Eq. (1). Figure 7.4.18 shows a plot of *maximal amplitude* (for the forced oscillations of any single floor) versus the *period* of the earthquake vibrations. The spikes correspond to the first six of the seven resonant frequencies. We see, for instance, that whereas an earthquake with period 2 (s) likely would produce destructive resonance vibrations in ~~the~~ building, it probably would be unharmed by an earthquake with period 2.5 (s). Different buildings have different natural frequencies of vibration, and so a given earthquake may demolish one building but leave untouched the one next door. This seeming anomaly was observed in Mexico City after the devastating earthquake of September ~~19, 1985.~~

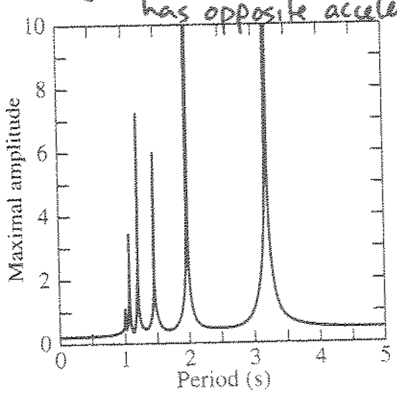

FIGURE 7.4.18. Resonance vibrations of a seven-story building—maximal amplitude as a function of period.

For your personal seven-story building to investigate, let the weight (in tons) of each story equal the largest digit of your student ID number and let k (in tons/ft) equal the smallest nonzero digit. Produce numerical and graphical results like those illustrated in Figs. 7.4.17 and 7.4.18. Is your building susceptible to likely damage from an earthquake with period in the 2- to 3-second range?

You ~~might like to begin by working manually the following~~ warm-up problems.

your project.
take $E = 4$ in
$= .25$ ft.

1. Find the periods of the natural vibrations of a building with two above-ground floors, each weighing 16 tons and with each restoring force being $k = 5$ tons/ft.
2. Find the periods of the natural vibrations of a building with three above-ground floors, with each weighing 16 tons and with each restoring force being $k = 5$ tons/ft.
3. Find the natural frequencies and natural modes of vibration of a building with three above-ground floors as in Problem 2, except that the upper two floors weigh 8 tons each instead of 16. Give the ratios of the amplitudes A, B, and C of oscillations of the three floors in the form $A : B : C$ with $A = 1$.
4. Suppose that the building of Problem 3 is subject to an earthquake in which the ground undergoes horizontal sinusoidal oscillations with a period of 3 s and an amplitude of 3 in. Find the amplitudes of the resulting steady periodic oscillations of the three above-ground floors. Assume the fact that a motion $E \sin \omega t$ of the ground, with acceleration $a = -E\omega^2 \sin \omega t$, produces an opposite inertial force $F = -ma = mE\omega^2 \sin \omega t$ on a floor of mass m.