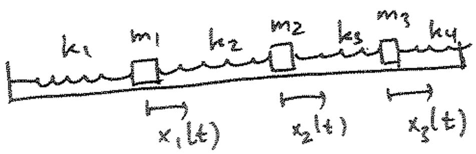
Math 2250-1
Mon 11/28

Continue §7.4, undamped spring systems.

Exercise 1   Derive the system of three 2nd order DE's for $x_1(t), x_2(t), x_3(t)$:



Write as $\quad M\vec{x}'' = K\vec{x}$

and $\quad \vec{x}'' = A\vec{x}$

Recall, for conservative systems as this with n masses we search for a solution space basis made of vector functions $\cos\omega t\,\vec{v}$, $\sin\omega t\,\vec{v}$. The solution space is $2n$-dimensional, so we'll need n independent vectors $\vec{v}$. In order for $\cos\omega t\,\vec{v}$ or $\sin\omega t\,\vec{v}$ to solve

$$\vec{x}'' = A\vec{x}$$

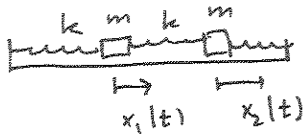
must have

$$-\omega^2\vec{v} = A\vec{v}$$

i.e. $\vec{v}$ is an eigenvector of A, with eigenvalue $\lambda = -\omega^2$ (so λ needs to be negative, and $\omega = \sqrt{-\lambda}$)

Exercise 2

Return to exercise 3 Wed., with the two mass, three spring configuration



$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Recompute the general soln (which we rushed through @ Wed.). This time first compute the eigenvalues and eigenvectors for

$$\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then use the fact that if $\vec{v}$ is an eigenvector of A with eigenvalue λ $(A\vec{v} = \lambda\vec{v})$, then $\vec{v}$ is also an eigenvector for the scalar multiple cA, except with eigenvalue $c\lambda$
$((cA)\vec{v} = cA\vec{v} = c\lambda\vec{v} \;\blacksquare)$.

[This reasoning will help w HW].

slow "in phase"

fast "out of phase"

ans

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left(c_1 \cos\sqrt{\tfrac{k}{m}}\,t + c_2 \sin\sqrt{\tfrac{k}{m}}\,t\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos\sqrt{\tfrac{3k}{m}}\,t + c_4 \sin\sqrt{\tfrac{3k}{m}}\,t\right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Calculations for a 2 mass– 3 spring system

Math 2250–1

November 28, 2011

The two mass, three spring system.

Data: Each mass is 50 grams. Each spring mass is 6 grams. (Remember, and this is a defect, our model assumes massless springs.) The springs are "identical", and a mass of 50 grams stretches the spring 15.6 centimeters. (We should recheck this, as well as testing the spring "Hookesiness"). Thus the spring constant is given by

```
> restart :
> Digits := 4 :
> solve(k·(.18) = .05·9.806, k);
```

$$2.724 \tag{1}$$

```
>
```

Let's time the two natural periods (which we discuss below):

Here's the model:

```
> with(LinearAlgebra) :
```

$$A := Matrix\left(2, 2, \left[-\frac{2 \cdot k}{m}, \frac{k}{m}, \frac{k}{m}, -\frac{2 \cdot k}{m}\right]\right);$$

$$A := \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \tag{2}$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} -\frac{k}{m} \\ -\frac{3k}{m} \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{3}$$

Predict the two natural periods from the model, and then run the experiment.

ANSWER: If you do the model correctly and my office data is close to our class data, you will come up with theoretical natural periods of close to .49 and .85 seconds. I predict that the actual natural periods are a little longer, especially for the slow mode. (In my office experiment I got periods of 0.51 and 0.91 seconds.) What happened?

EXPLANATION: The springs actually have mass, equal to 6 grams each. This is almost on the same order of magnitude as the yellow masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more–complicated situation, using the ideas of total energy discussed in section 5.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator (oct26.pdf), assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an A–matrix the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{6\,k}{6\,m + 5\,m_s}$$

$$\lambda_2 = -\frac{6\,k}{2\,m + m_s}.$$

(Hints: the "M" matrix is not diagonal but the "K" matrix is the same.)

If you use these values, then you get period predictions which might be closer to our experiment.

```
> m := .05;
  ms := .006;
  k := 2.724;
  Omega1 := sqrt(6·k/(6·m + 5·ms));
  Omega2 := sqrt(6·k/(2·m + ms));
  T1 := evalf(2·Pi/Omega1);
  T2 := evalf(2·Pi/Omega2);
```

$$m := 0.05$$

$$ms := 0.006$$

$$k := 2.724$$

$$\Omega 1 := 7.038$$

$$\Omega 2 := 12.42$$

$$T1 := 0.8930$$

$$T2 := 0.5059 \qquad (4)$$

```
>
```

Challenge: If you can construct (and explain to me in my office) a correct derivation of the eigenvalues /eigenvectors I claim above, by taking the spring masses into account, then you can either substitute your derivation for the section 7.4 Maple exploration in next week's homework, or get 10 bonus points on the final exam. This is a challenging challenge, but it's definitely doable!

Forced oscillations (still undamped)

$$M \vec{x}'' = K\vec{x} + \vec{F}(t)$$

e.g. sinusoidal @ angular frequency ω

$$* \qquad \vec{x}'' = A\vec{x} + \cos\omega t\, \vec{F}_0$$

$$(A = M^{-1}K; \quad \cos\omega t\, \vec{F}_0 = M^{-1}\vec{F}(t)).$$

the general soltn will be

$$\vec{x} = \vec{x}_p + \vec{x}_H$$

already understand

as long as ω is not one of the natural angular frequencies, use undetermined coeff's.

$$\vec{x}_p = \cos\omega t\, \vec{c} \ ; \quad \vec{c} \text{ unknown:}$$

Plug $\vec{x}_p$ into $*$:

$$-\omega^2 \cancel{\cos\omega t}\, \vec{c} = \cancel{\cos\omega t}\, A\vec{c} + \cancel{\cos\omega t}\, \vec{F}_0$$

$$-\vec{F}_0 = A\vec{c} + \omega^2\vec{c}$$

$$-\vec{F}_0 = (A + \omega^2 I)\vec{c}$$

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}_0)$$

eval of A

as long as $-\omega^2 \neq \lambda$, since in that case matrix is singular

Continue with our example, assume $\frac{k}{m} = 1$, and force the second mass:

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 3\cos\omega t \end{bmatrix}}_{\cos\omega t \begin{bmatrix} 0 \\ 3 \end{bmatrix}.}$$

$$\vec{x}_p = \cos\omega t \, \vec{c}.$$

<u>Exercise 3</u> Find the particular solution, and general soltn, assuming $\omega \neq 1, \sqrt{3}$

<u>soltn</u>

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\cos\omega t \begin{bmatrix} \frac{3}{(\omega^2-3)(\omega^2-1)} \\ \frac{6-3\omega^2}{(\omega^2-3)(\omega^2-1)} \end{bmatrix}}_{\substack{\text{with slight damping} \\ \approx \vec{x}_{sp}}} + \underbrace{C_1\cos(t-\alpha_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\cos(\sqrt{3}t-\alpha_2)\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\substack{\text{with slight damping} \\ \approx \vec{x}_{tr}}}$$

Practical resonance calculations for class example, November 28.

> *restart* :

> *with*(*LinearAlgebra*) :

> $A := Matrix(2, 2, [-2, 1, 1, -2]);$

$$A := \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \qquad (1)$$

> $F0 := Vector([0, 3]);$

$$F0 := \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad (2)$$

> $Iden := IdentityMatrix(2);$

$$Iden := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (3)$$

> $c := \omega \rightarrow \left(A + \omega^2 \cdot Iden\right)^{-1} . (-F0);$

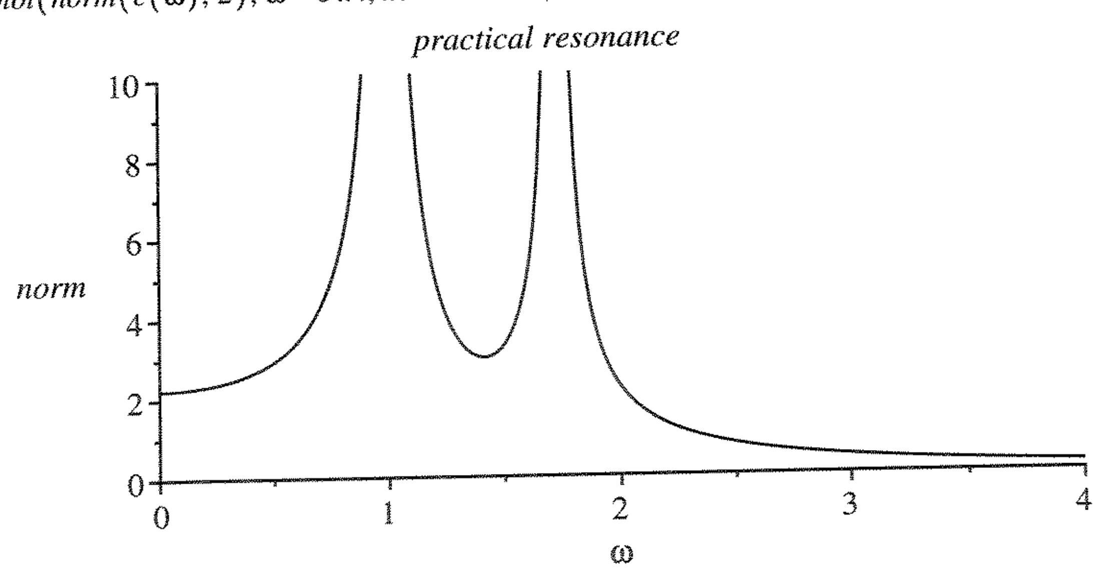
$$c := \omega \rightarrow Typesetting\text{:-}delayDotProduct\left(\frac{1}{A + \omega^2\,Iden}, -F0\right) \qquad (4)$$

> $c(\omega);$

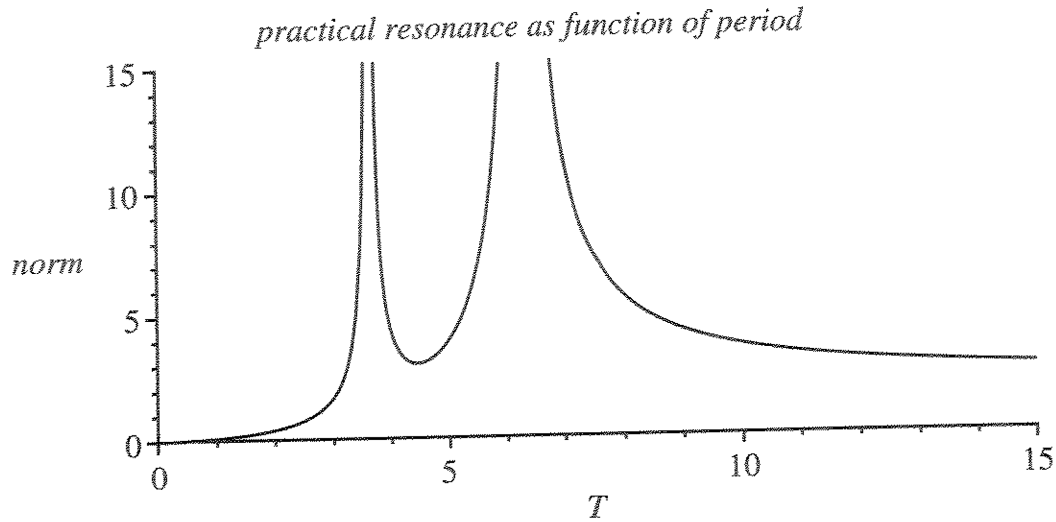
$$\begin{bmatrix} \dfrac{3}{3 - 4\,\omega^2 + \omega^4} \\ -\dfrac{3\left(-2 + \omega^2\right)}{3 - 4\,\omega^2 + \omega^4} \end{bmatrix} \qquad (5)$$

> *with*(*plots*) :

> $plot(norm(c(\omega), 2), \omega = 0..4, norm = 0..10, color = black, title = \text{'practical resonance'});$



> $plot\left(norm\left(c\left(\frac{2\cdot \mathrm{Pi}}{T}\right), 2\right), T = 0..15, norm = 0..15, color = black, title = \text{'practical resonance as function of period'}\right);$



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