Math 2250-1
Wed 11/23

First:
* do glucose/insulin model in Tuesday notes

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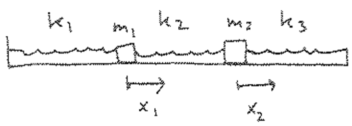
§7.4 undamped spring systems

In your HW you modeled this spring system (no damping):



IVP: $x_1(0) = b_1$ $\quad x_2(0) = c_1$
$x_1'(0) = b_2$ $\quad x_2'(0) = c_2$

$$m_1 x_1'' = k_2(x_2 - x_1) - k_1 x_1$$
$$m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -k_1-k_2 & k_2 \\ k_2 & -k_2-k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$M\vec{x}'' = K\vec{x}$$

is the general equation with n masses & up to n+1 springs in series

$\Rightarrow \boxed{\vec{x}'' = A\vec{x}}$ where $A = M^{-1}K$ is the "acceleration" matrix

in our example,

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2-k_3}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let's try $\vec{x}(t) = e^{\lambda t}\vec{v}$: (It's worked so well for 1^{st} order systems!)

$$\vec{x}' = \lambda e^{\lambda t}\vec{v}$$
$$\vec{x}'' = \lambda^2 e^{\lambda t}\vec{v}$$
$$A\vec{x} = e^{\lambda t}A\vec{v}$$

So now, we'd need

$$A\vec{v} = \lambda^2\vec{v}$$

And, given that this is a conservative system (no drag), what sort of λ's do we expect: Total energy must remain constant!

$\lambda = a + bi \quad \vec{v} = \vec{u} + i\vec{w}$

$$e^{\lambda t}\vec{v} = e^{at}(\cos bt + i\sin bt)(\vec{u} + i\vec{v})$$
$$= e^{at}(\cos bt\,\vec{u} - \sin bt\,\vec{v})$$
$$+ i\,e^{at}(\sin bt\,\vec{u} + \cos bt\,\vec{v})$$

TE = const
$\Rightarrow a = 0$
$\Rightarrow \lambda$ purely imaginary
$\Rightarrow b =$ angular frequency

so $\lambda = bi$

rewrite it as

$\lambda = \omega i$

$$A\vec{v} = \lambda^2 \vec{v} = -\omega^2 \vec{v}$$

$$e^{\lambda t} = e^{i\omega t}\vec{v} = \cos\omega t\,\vec{v} + i\sin\omega t\,\vec{v}$$

Exercise 1 Check that if we go directly to the guess

$$\vec{x}(t) = \cos\omega t\,\vec{v} \quad \text{or} \quad \vec{x}(t) = \sin\omega t\,\vec{v}$$

as solutions to

$$\vec{x}''(t) = A\vec{x}$$

Then we must have $A\vec{v} = -\omega^2\vec{v}$. In other words, $\omega = \sqrt{-\lambda}$, where $A\vec{v} = \lambda\vec{v}$ is the eigenvalue - eigenvector eqn.

our hope : If A is diagonalizable we will get n independent eigenvectors.
Each eigenvector $\vec{v}$ yields two solns $\cos\omega t\,\vec{v}$, $\sin\omega t\,\vec{v}$ $(\omega = \sqrt{-\lambda})$

$\leadsto$ get $2n$ solutions, which is the dimension of the solution space!

Exercise 3 Let $k_1 = k_2 = k_3 = k$

$m_1 = m_2 = m$, so that the system on page 1 reduces to

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the 4-dim'l solution space!

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

<u>answer</u>: $\vec{x}(t) = \underbrace{\left(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t\right)}_{C_1 \cos(\omega_1 t - \alpha_1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{\left(c_3 \cos\omega_2 t + c_4 \cos\omega_2 t\right)}_{C_2 \cos(\omega_2 t - \alpha_2)} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

slow, in-phase mode (masses oscillate in parallel)

faster, out of phase mode (masses oscillate in opposition)

Calculations for a 2 mass– 3 spring system

Math 2250–1

November 23, 2011

The two mass, three spring system.

Data: Each mass is 50 grams. Each spring mass is 10 grams. (Remember, and this is a defect, our model assumes massless springs.) The springs are "identical", and a mass of 50 grams stretches the spring 15.6 centimeters. (We should recheck this, as well as testing the spring "Hookesiness"). Thus the spring constant is given by

```
> Digits := 4 :
> solve(k*.156=.05*9.806,k);
```

$$3.143 \tag{1}$$

Let's time the two natural periods (which we discuss below):

Here's the model:

```
> with(LinearAlgebra):
> A:=Matrix(2,2,[-2*k/m, k/m,k/m,-2*k/m]);
    #this should be the "A" matrix you get for
    #our two-mass, three-spring system.
```

$$A := \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \tag{2}$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} -\frac{3k}{m} \\ -\frac{k}{m} \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3}$$

Predict the two natural periods from the model:

ANSWER: If you do the model correctly and my office data is close to our class data, you will come up with theoretical natural periods of close to .46 and .79 seconds. I predict that the actual natural periods are a little longer, especially for the slow mode. (In my office experiment I got periods of 0.482 and 0.855 seconds.) What happened?

EXPLANATION: The springs actually have mass, equal to 10 grams each. This is almost on the same order of magnitude as the yellow masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more–complicated situation, using the ideas of total energy discussed in section 5.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator (mar11.pdf), assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an A–matrix the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{6\,k}{6\,m + 5\,m_s}$$

$$\lambda_2 = -\frac{6\,k}{2\,m + m_s}.$$

(Hints: the "M" matrix is not diagonal but the "K" matrix is the same.)

If you use these values, then you get period predictions

```
> m:=.050;
  ms:=.010;
  k:=3.143;
  Omega1:=sqrt(6*k/(6*m+5*ms));
  Omega2:=sqrt(6*k/(2*m+ms));
  T1:=evalf(2*Pi/Omega1);
  T2:=evalf(2*Pi/Omega2);
```

$$m := 0.050$$

$$ms := 0.010$$

$$k := 3.143$$

$$\Omega 1 := 7.340$$

$$\Omega 2 := 13.09$$

$$T1 := 0.8559$$

$$T2 := 0.4801 \tag{4}$$

of .856 and .480 seconds per cycle. Is that closer?

Challenge: If you can construct (and explain to me in my office) a correct derivation of the eigenvalues /eigenvectors I claim above, by taking the spring masses into account, then you can either substitute your derivation for the section 7.4 Maple exploration in next week's homework, or get 10 bonus points on the final exam. This is a challenging challenge, but it's definitely doable!