Math 2250-1
Tues 11/22

HW problem session this afternoon
4-5:30 pm
WEB L102

§7.3: complex eigenvalues and eigenvectors; applications

- we were almost done with Exercise 2 from Monday's notes.
- Exercise 3 introduces complex eigenvalues and eigenvectors, and their use in 1^{st} order systems of DE's.

The general overview for $\lambda = a+bi$ complex is below:

$\vec{x}_H$ for $\frac{d\vec{x}}{dt} = A\vec{x}$ when
- A is a real coeff matrix
- $\lambda = a+bi$ complex eval
  $\vec{v} = \vec{\alpha} + i\vec{\beta}$ complex evect.

Then
$\vec{z}(t) = e^{\lambda t}\vec{v}$ is a complex solution, since $\frac{d\vec{z}}{dt} = \lambda e^{\lambda t}\vec{v}$
and $A\vec{z} = Ae^{\lambda t}\vec{v} = e^{\lambda t}\lambda\vec{v}$ (still works, just as for real λ.)

From this complex solution we extract 2 linearly independent solutions:

$$\vec{x}(t) = Re(e^{\lambda t}\vec{v})$$
$$\vec{y}(t) = Im(e^{\lambda t}\vec{v})$$

<u>check</u> $\vec{z}(t) = \vec{x}(t) + i\vec{y}(t)$
$\Rightarrow \vec{z}\,'(t) = \vec{x}\,'(t) + i\vec{y}\,'(t)$
$A\vec{z} = A\vec{x}(t) + iA\vec{y}(t)$

since $z' = Az$
$\Rightarrow \vec{x}\,' + i\vec{y}\,' = A\vec{x} + iA\vec{y}$

since real & imag parts must agree for complex #'s to be equal, deduce

$\vec{x}\,' = A\vec{x}$
& $\vec{y}\,' = A\vec{y}$

(one can check linear independence).

(You will get the same two sol'ns, up to sign, from $e^{(a+bi)t}\vec{v}$ and $e^{(a-bi)t}\bar{\vec{v}}$)

Modeling with first order systems of DE's

Math 2250–1
Tuesday Nov. 22, 2011

Exercise 1: Tanks: This is example 2 from our text, page 422. (For most real input–output models the eigenvalues and eigenvectors would be better expressed using decimal approximations; examples with algebraically nice eigenvalues are hard to come by but good for conceptual purposes.) We have a cascade of three tanks, see figure 7.3.2 page 422, with pure water flowing into tank 1, and well–mixed solutions flowing from tank 1 to tank 2, from tank 2 to tank 3, and from tank 3 to our of the system. The tank volumes are $V_1 = 20$, $V_2 = 40$, $V_3 = 50$ gallons. All the water flow rate are $r = 10$ g/min. The initial amounts of salt are $x_1(0) = 15, x_2(0) = 0, x_3(0) = 0$ pounds. (You actually had a 2–tank cascade way back in Chapter 1, which you solved in steps, without using systems of DEs.)

1a) Sketch a diagram and show that the initial value problem for this system is

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix}$$

1b) Find a basis for the solution space, using a basis made out of solutions $\underline{\mathbf{x}}(t) = e^{\lambda \cdot t}\underline{\mathbf{v}}$. Then solve the initial value problem. You can check your eigenvalue–eigenvector work with:

```
> with(LinearAlgebra):
> A:=Matrix(3,3,[-1/2, 0, 0, 1/2, -1/4, 0, 0, 1/4, -1/5]):
  Eigenvectors(A);
```

$$\begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} 0 & \frac{3}{5} & 0 \\ 0 & -\frac{6}{5} & -\frac{1}{5} \\ 1 & 1 & 1 \end{bmatrix} \qquad (1)$$

You should get

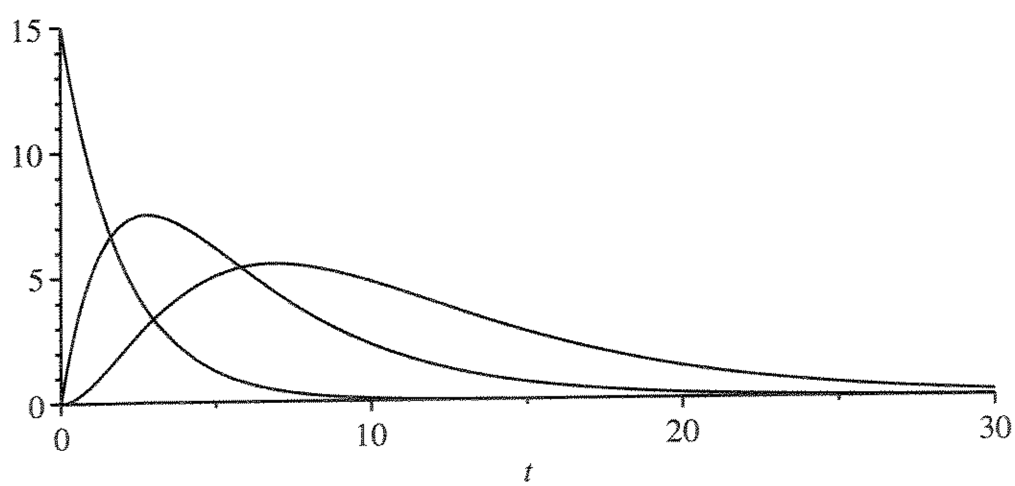
$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 5\,e^{-0.5\,t}\begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + 30\,e^{-0.25\,t}\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125\,e^{-0.2\,t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can plot the three solute amount vs. time to see what is going on:

```
> x1:=t->15*exp(-.5*t):
  x2:=t->-30*exp(-.5*t)+30*exp(-.25*t):
  x3:=t->25*exp(-.5*t)-150*exp(-.25*t)+125*exp(-.2*t):
```

```
> plot({x1(t),x2(t),x3(t)},t=0..30,color=black,
  title=`salt contents in each tank`);
```

salt contents in each tank



Example 2: Glucose–insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let G(t) be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let H(t) be the excess insulin concentration at time t. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} \frac{dG}{dt} \\ \frac{dH}{dt} \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

Explain (understand) the signs of the matrix coefficients:

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

```
> with(plots):
> A:=Matrix(2,2,[-1/10,-4/10,1/10,-1/10]);
```

$$A := \begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix} \quad (2)$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} -\frac{1}{10} + \frac{1}{5}\,\mathrm{I} \\ -\frac{1}{10} - \frac{1}{5}\,\mathrm{I} \end{bmatrix}, \begin{bmatrix} 2\,\mathrm{I} & -2\,\mathrm{I} \\ 1 & 1 \end{bmatrix} \quad (3)$$

```
>
```

Can you get the same eigenvalues and eigenvectors? Notice that Maple writes a capital I for the square root of −1, i. Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

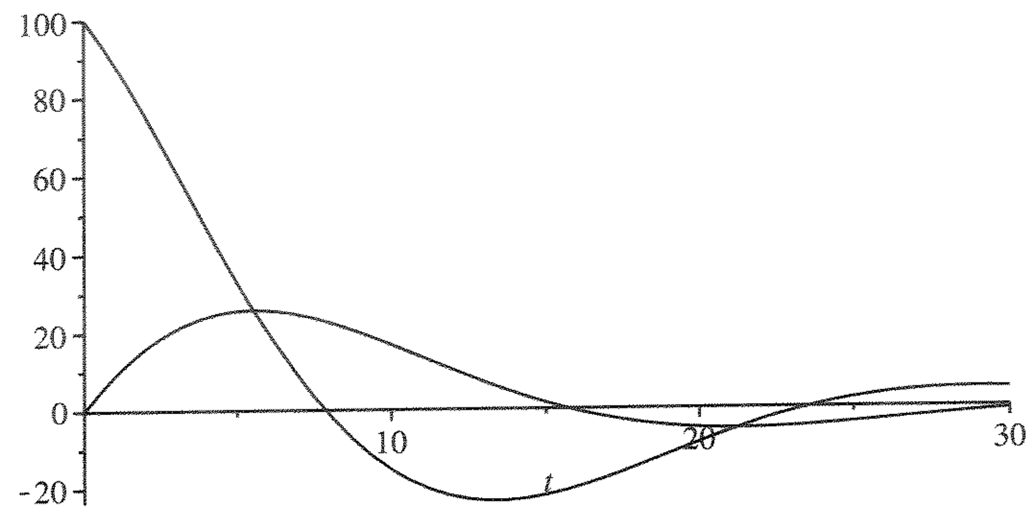
Solve the initial value problem.

Here are some pictures to help understand what the model is predicting ... you could also construct these graphs using pplane.

(1) Plots of glucose vs. insulin, at time t hours later:

```
> G:=t->100*exp(-.1*t)*cos(.2*t):
  H:=t->50*exp(-.1*t)*sin(.2*t):
> plot({G(t),H(t)},t=0..30,color=black,title=
  `Excess Glucose and Insulin concentrations`);
```

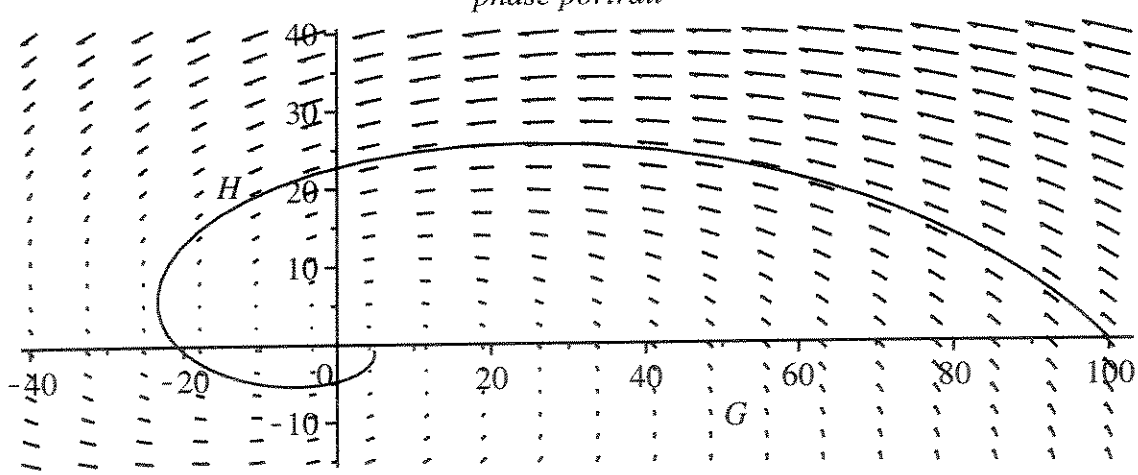
Excess Glucose and Insulin concentrations



2) A phase portrait of the glucose–insulin system:

```
> pict1:=fieldplot([-.1*G-.4*H,.1*G-.1*H],G=-40..100,H=-15..40):
  soltn:=plot([G(t),H(t),t=0..30],color=black):
  display({pict1,soltn},title=`Glucose vs Insulin
  phase portrait`);
```

Glucose vs Insulin
phase portrait




```
>
```