Math 2250-1
Mon 11/21
§7.1-7.3

HW session Tues 4-5:30
location TBA

- Finish Friday's notes – we were talking about how systems of differential equations arise in modeling multicomponent dynamical systems; about how every DE or system of DE's is equivalent to a 1st order system, and how the IVP's for which we expect existence and uniqueness of solutions seem to correspond to a 1st order system of DE's IVP:

$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = \vec{F}(t, \vec{x}(t)) & \leftarrow \text{know "velocity", based on current location \& time} \\ \vec{x}(t_0) = \vec{x}_0 & \leftarrow \text{know where to start} \end{cases}$$

- pages 4-5 Fri
- Then:

General result

$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

If $\vec{f}$ is differentiable in its variables the there exists a unique sol'n to IVP, defined on some interval $(t_0 - \delta, t_0 + \delta)$

Linear systems of DE's result

$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = A(t)\,\vec{x} + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

usually constant in 2250

- $A(t)$ is an $n \times n$ matrix fcn of t
- system is homogeneous if $\vec{f}(t) \equiv 0$ (else inhomogeneous)

If $A(t)$ and $\vec{f}(t)$ are continuous on an interval containing t_0, then there exists a unique sol'n to IVP, defined on the entire interval.

Numerical solutions: Euler: time step $= h$

$$t_j = t_0 + jh$$
$$\vec{x}_{j+1} = \vec{x}_j + h \cdot \vec{f}(t_j, \vec{x}_j)$$

$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

also improved Euler & Runge-Kutta

Systems of linear 1^{st} order DE's theory

(this should feel familiar)

1. The operator $L(\vec{x}(t)) := \vec{x}'(t) - A(t)\,\vec{x}(t)$

   is linear, i.e.

   $$L(\vec{x}(t) + \vec{z}(t)) = L(\vec{x}(t)) + L(\vec{z}(t))$$
   $$L(c\,\vec{x}(t)) = c\,L(\vec{x}(t)).$$

   check!

2. Thus the general solution to

   $$\vec{x}'(t) - A(t)\,\vec{x}(t) = \vec{f}(t)$$

   is $\vec{x}(t) = \vec{x}_p(t) + \vec{x}_H(t)$

   where $\vec{x}_p(t)$ is any single (particular) solution to the <u>nonhomogeneous</u> problem $(\vec{f} \neq \vec{0})$ and $\vec{x}_H(t)$ is the general solution to the homogeneous system

   $$\vec{x}'(t) - A(t)\,\vec{x}(t) = 0$$

   (which we often write as $\vec{x}' = A\vec{x}$)

3. The solution space to the homogeneous system $\vec{x}' = A\vec{x}$, i.e. to

   $$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & \\ \vdots & & & \\ a_{n1}(t) & \cdots & & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

   on the interval of $t \in I$ is n-dimensional.

   In fact, let $\vec{X}_1(t), \vec{X}_2(t) \ldots \vec{X}_n(t)$ be n solutions so that the <u>Wronskian matrix</u> at t_0,

   $$W(\vec{X}_1, \ldots \vec{X}_n)(t_0) := \left[\, \vec{X}_1(t_0) \,\Big|\, \vec{X}_2(t_0) \,\Big|\, \cdots \,\Big|\, \vec{X}_n(t_0) \,\right]$$

   has an inverse (i.e. $\det \neq 0$, $\operatorname{rref} = I$)

   (called the Wronskian)

   [the existence theorem says such solutions exist].

   Then the solution to

   $$\begin{cases} \vec{x}' = A(t)\,\vec{x} \\ \vec{x}(t_0) = \vec{b} \end{cases}$$

   is $\vec{x}(t) = c_1\vec{X}_1(t) + c_2\vec{X}_2(t) + \cdots + c_n\vec{X}_n(t)$

   where $$\left[\, \vec{X}_1(t_0) \,\Big|\, \vec{X}_2(t_0) \,\Big|\, \cdots \,\vec{X}_n(t) \right] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

   Thus $\vec{X}_1, \vec{X}_2, \ldots \vec{X}_n$
   - span the soln space and
   - are linearly independent.

   so are a basis!

§7.3 Eigenvalue-eigenvector method for solving the homogeneous system

$\vec{x}' = A\vec{x}$ when A is a constant matrix

We look for a basis of the solution space made of $\vec{x}(t)$ of the form $e^{\lambda t}\vec{v}$: ($\vec{v}$ a constant vector)

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

$$\Rightarrow \vec{x}'(t) = \lambda e^{\lambda t}\vec{v}$$

$$A\vec{x} = A(e^{\lambda t}\vec{v}) = e^{\lambda t}(A\vec{v})$$

so we need

$$\lambda \cancel{e^{\lambda t}}\vec{v} = \cancel{e^{\lambda t}} A\vec{v}$$

$$\lambda\vec{v} = A\vec{v}$$

: $\vec{v}$ should be an eigenvector of A, and λ should be its eigenvalue.

<u>If</u> there is a basis of $\mathbb{R}^n$ made of eigenvectors of A, i.e. if A is diagonalizable, then we can construct

$$\vec{X}_1(t) = e^{\lambda_1 t}\vec{v}_1,\ \vec{X}_2(t) = e^{\lambda_2 t}\vec{v}_2, \ldots\ \vec{X}_n(t) = e^{\lambda_n t}\vec{v}_n$$

$$W(\vec{X}_1, \vec{X}_2, \ldots \vec{X}_n)\Big|_{t=0} = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right]$$

is non-singular, i.e. $\vec{X}_1, \vec{X}_2, \ldots \vec{X}_n$ are a basis for the solution space.

<u>Exercise 1</u> Use this method to find the general solution to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

, and then solve the IVP with $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Exercise 2 Check for solutions $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

the function $x(t)$ satisfies the overdamped DE

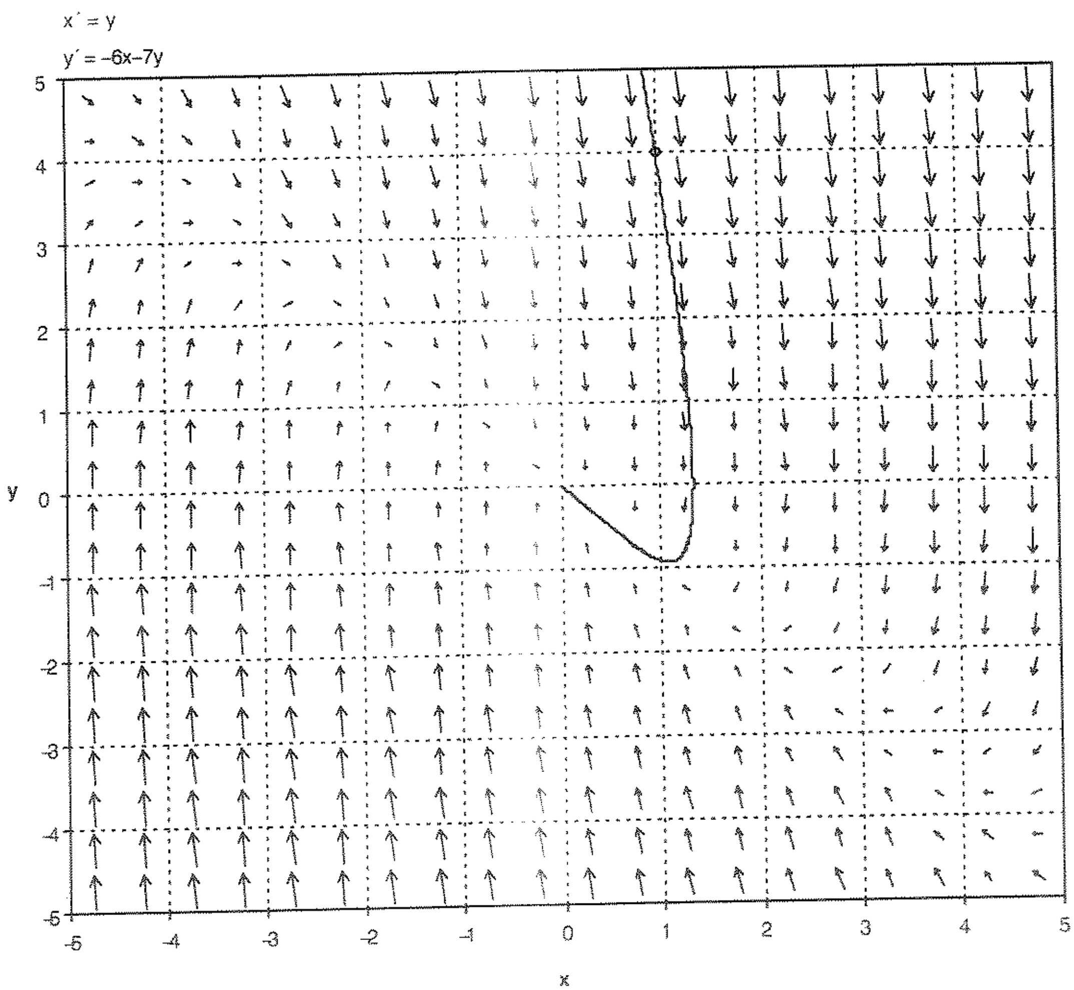
$$x'' + 7x' + 6x = 0$$

and our IVP sol'n satisfies $x(0) = 1$, $x'(0) = 4$.
Interpret the phase portrait below. Can you find the eigenvector directions from the tangent vector field?

Also, compare characteristic polynomials and Wronskians for

Chapter 5 $\quad x'' + 7x' + 6x = 0$ $\qquad$ Chapter 7 $\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

$$\left(\begin{array}{l} x' = y \\ y' = -6x - 7y \end{array} \right)$$



Exercise 3 (If time). Find the general solution to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

using complex eigenvalues and eigenvectors (?!)