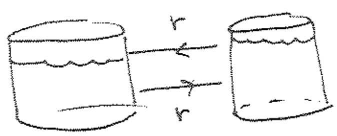
Math 2250-1
Fri 11/18

§7.1 Systems of differential equations :
to model multi-component dynamical systems!

Examples

1. input-output models:



$V_1 = 100$ g $\qquad V_2 = 50$ g

$x_1(t) \qquad x_2(t)$

solute amounts, changing in time

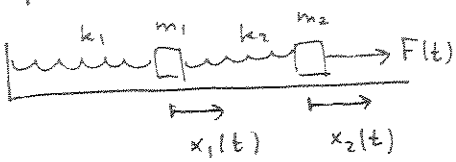
suppose $r = 100$ g/hour

Exercise 1 : Find differential equations for $x_1(t), x_2(t)$, using input-output modeling.
If $x_1(0) = b_1$, $x_2(0) = b_2$ are initial solute amounts write down the initial value problem you would expect to have a unique soltn.

ans (in vector-matrix form):

$$\text{IVP} \begin{cases} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{cases}$$

(2) coupled masses & springs (or inductors and capacitors).



displacements from equilibrium.

Exercise 2 : Use Newton's second law to derive a system of two 2nd order DE's for $x_1(t)$ and $x_2(t)$ in the configuration above. What IVP do you expect yields unique solutions?

Exercise 3 Let $m_1 = 2$ $\quad k_1 = 4$ $\quad F(t) = 40 \sin 3t$
$m_2 = 1$ $\quad k_2 = 2$

Show that the system above can be written in vector-matrix form as

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 40 \sin 3t \end{bmatrix}$$

3. Converting systems of higher order DE's into equivalent (larger) systems of 1^{st} order DE's is always possible. This turns out to be important conceptually and also for computational reasons. You do this by introducing extra functions for intermediate derivatives:

page 2 IVP:

$$\text{IVP}_1 \begin{cases} x_1'' = -3x_1 + x_2 \\ x_2'' = 2x_1 - 2x_2 + 40\sin 3t \\ x_1(0) = b_1 \qquad x_2(0) = c_1 \\ x_1'(0) = b_2 \qquad x_2'(0) = c_2 \end{cases}$$

if $x_1(t), x_2(t)$ solve IVP_1

let $y_1(t) = x_1'(t)$

$y_2(t) = x_2'(t)$.

Then x_1, y_1, x_2, y_2 solve a 1^{st} order IVP:

$$\text{IVP}_2 \begin{cases} x_1' = y_1 \\ y_1' = -3x_1 + x_2 \\ x_2' = y_2 \\ y_2' = 2x_1 - 2x_2 + 40\sin 3t \\ x_1(0) = b_1 \\ y_1(0) = b_2 \\ x_2(0) = c_1 \\ y_2(0) = c_2 \end{cases}$$

also, if x_1, y_1, x_2, y_2 solve IVP_2, then $x_1(t), x_2(t)$ solve IVP_1

Check this!!

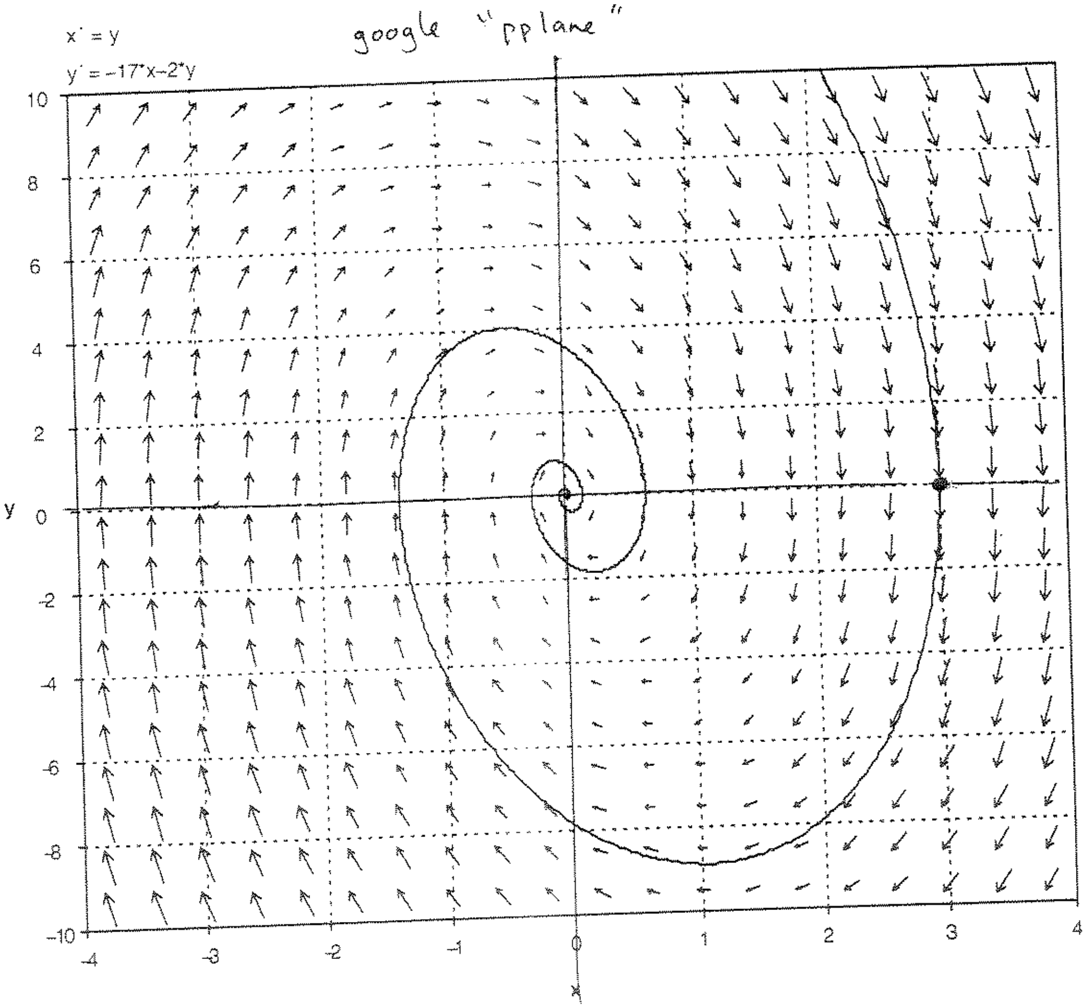
Exercise 4 Convert the IVP

$$\begin{cases} x'' + 2x' + 17x = 0 \\ x(0) = 4 \\ x'(0) = 0 \end{cases}$$

into an equivalent 1^{st} order system IVP. Solve the underdamped 2^{nd} order IVP in order to deduce a sol'n to the 1^{st} order system IVP. Is your solution consistent with the image of the parametric curve shown below?

Also, compare the characteristic polynomial $p(r)$ for the 2^{nd} order DE to the characteristic polynomial $\det(A - \lambda I)$ for the matrix in the 1^{st} order system ?!

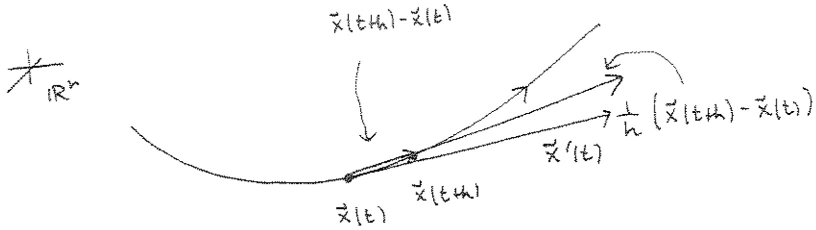
google "pplane"



Geometric interpretation of 1^{st} order system of DE's:

IVP $\begin{cases} \dfrac{d\vec{x}}{dt} = \vec{F}(t,\vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$

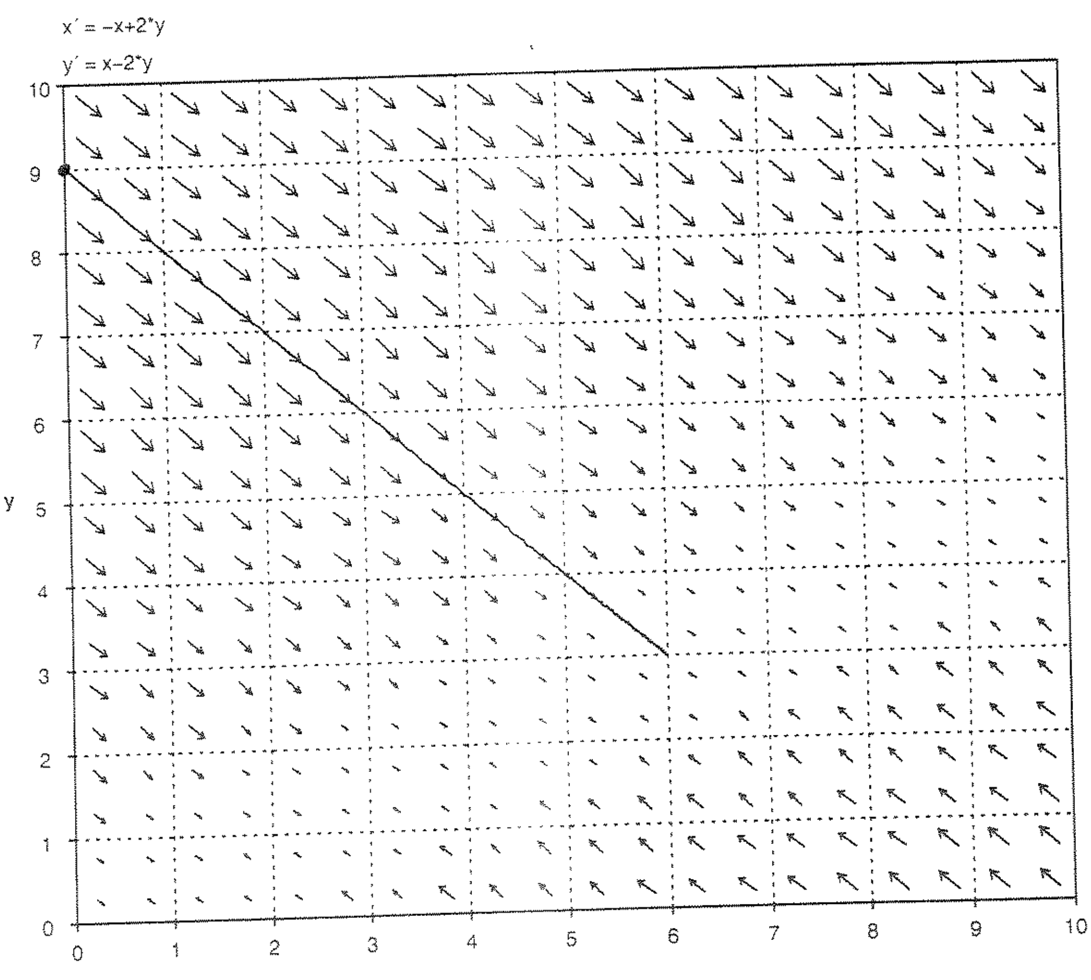
- Recall, if $\vec{x}(t)$ is parametric curve in space (think particle position at time t) then $\vec{x}'(t)$ is the tangent (or velocity) vector:



So the IVP says you know where you start $(\vec{x}(t_0)=\vec{x}_0)$, and you know your velocity vector (depending on time & location) $\rightarrow$ so you expect a unique sol'n

page 1 example!

$\begin{cases} x_1'(t) = -x_1 + 2x_2 \\ x_2'(t) = x_1 - 2x_2 \\ x_1(0) = 0 \\ x_2(0) = 9 \end{cases}$



notice, in this example, your "particle" location $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ is really the solute amounts in tanks 1 & 2.

What is

$\lim_{t\to\infty} x_1(t)$?

$\lim_{t\to\infty} x_2(t)$?