Math 2250-1
Wed 11/16

§6.1-6.2 eigenvalues, eigenvectors, diagonalizability. ← Friday quiz on 6.1-6.2

Recall from Tuesday,

Def. If $A_{n\times n}$ and $A\vec{v} = \lambda\vec{v}$ for some scalar λ and some vector $\vec{v} \neq \vec{0}$, then $\vec{v}$ is called an eigenvector of A, and λ is called an eigenvalue of A, associated the the eigenvector $\vec{v}$.

- Eigenvalues & eigenvectors are important in a variety of applications.
- For example, if we can construct an $\mathbb{R}^n$ basis made out of eigenvectors of A, then we get a good understanding of the transformation $T(\vec{x}) = A\vec{x}$.

At the end of class yesterday we were working on a 3×3 example:

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

step 1 set $\det(A-\lambda I)=0$ to find eigenvalues

long way:

(did a shortcut yesterday)

$$\begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix} \overset{\text{top row}}{=} 4-\lambda \begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} - -2 \begin{vmatrix} 2 & 1 \\ 2 & 3-\lambda \end{vmatrix} + 1 \begin{vmatrix} 2 & -\lambda \\ 2 & -2 \end{vmatrix}$$

$$= (4-\lambda)(\lambda^2-3\lambda+2) + 2(6-2\lambda-2) + 1(-4+2\lambda)$$

$$= \lambda^3(-1) + \lambda^2(4+3) + \lambda(-12-2-4+2) + 1(8+8-4)$$

$$p(\lambda) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$$

↑ integer λ divide 12

$p(2) = -8+28-32+12 = 0$

so $\lambda = 2$ is a root

$$\begin{array}{r|l} & -\lambda^2 + 5\lambda - 6 \\ \hline \lambda-2 & -\lambda^3 + 7\lambda^2 - 16\lambda + 12 \\ & -\lambda^3 + 2\lambda^2 \\ \hline & 5\lambda^2 - 16\lambda \\ & 5\lambda^2 - 10\lambda \\ \hline & -6\lambda + 12 \\ & -6\lambda + 12 \\ \hline & 0 \end{array}$$

so $p(\lambda) = (\lambda-2)(-\lambda^2+5\lambda-6)$

$= -(\lambda-2)(\lambda^2-5\lambda+6)$

$= -(\lambda-2)(\lambda-2)(\lambda-3)$

$p(\lambda) = -(\lambda-2)^2(\lambda-3)$

<u>step 2</u> for each eigenvalue from step 1, find a basis for the eigenspace.

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

we did $\lambda = 2$: (solutions to $A\vec{v} - 2\vec{v} = \vec{0}$)

$A - 2I$:
$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

front-solve:
$v_1 = t$
$v_2 = q$
$v_3 = -2t + 2q$

$$\vec{v} = t\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + q\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

so $E_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$

↑
the $\lambda = 2$ eigenspace

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we are here:
recall that homogeneous solutions correspond to column dependencies.
from rref$(A-2I)$ we know soln space to $(A-2I)\vec{v} = \vec{0}$ is 2-dimensional

since $1 \cdot \text{col}_2 + 2 \cdot \text{col}_3 = \vec{0} \rightarrow \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$1 \cdot \text{col}_1 - 2\,\text{col}_3 = \vec{0} \rightarrow \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

is a quick way to get a basis for E_2.
is there a different easy dependency that would yield alternate bases for E_2?

<u>Exercise 1</u> Find a basis for the $\lambda = 3$ eigenspace of A

Notice that

$$\left\{ \underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}}_{\substack{\lambda=2\\ \text{eigenbasis}}}, \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\lambda=3} \right\}$$

is a basis for $\mathbb{R}^3$. This is good for algebra <u>and</u> geometry <u>and</u> differential eqtns related to the matrix A

Algebra:

$$\underbrace{\begin{bmatrix}4 & -2 & 1\\2 & 0 & 1\\2 & -2 & 3\end{bmatrix}}_{A}\underbrace{\left[\begin{array}{c|c|c}0 & 1 & 1\\1 & 0 & 1\\2 & -2 & 1\end{array}\right]}_{P} = \left[2\begin{bmatrix}0\\1\\2\end{bmatrix}, 2\begin{bmatrix}1\\0\\-2\end{bmatrix}, 3\begin{bmatrix}1\\1\\1\end{bmatrix}\right]$$

$$= \underbrace{\begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\2 & -2 & 1\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}}_{D}$$

cols are evects

diagonal matrix; entries are evals

$$AP = PD$$

<u>algebra</u> application

$$AP = PD$$

$$P^{-1}AP = D$$

or $\quad A = PDP^{-1}$

So $\quad A^{100} = (PDP^{-1})(PDP^{-1})(\quad)\cdots(PDP^{-1})$

$$= PD^{100}P^{-1}, \quad \text{(because all the interior } P^{-1}P = I)$$

$$A^{100} = P\begin{bmatrix}2^{100} & 0 & 0\\0 & 2^{100} & 0\\0 & 0 & 3^{100}\end{bmatrix}P^{-1}$$

because products of diagonal matrices are diagonal.

- only need to do 2 matrix multiplications to compute A^{100} (or A^n, any integer n)

<u>Definition</u>

for this reason, if there exists an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis made out of eigenvectors, then A is called <u>diagonalizable</u>

- in this case let $P_{n\times n} = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right]$ $\quad A\vec{v}_j = \lambda_j \vec{v}_j$.

then $A\left[\vec{v}_1 \middle| \cdots \middle| \vec{v}_n\right] = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right]\begin{bmatrix}\lambda_1 & 0 & 0\\0 & \lambda_2 & \vdots\\0 & \vdots & \lambda_n\end{bmatrix}$

$$AP = PD$$

$$P^{-1}AP = D$$

$$A = PDP^{-1}$$

unfortunately, not every matrix is diagonalizable

<u>Exercise 3</u> Show that $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalizable

ans $|B - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = -(\lambda-2)^2(\lambda-3)$

(for an upper or lower triangular matrix, the eigenvalues are exactly the diagonal entries)

$\lambda = 2$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$v_1 = t$
$v_2 = 0$
$v_3 = 0$

$\vec{v} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Eigenspace basis
$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

this eigenspace is called "defective", because

$p(\lambda) = -(\lambda-2)^2(\lambda-3)$

$\lambda = 2$ has algebraic multiplicity 2, but its eigenspace is only 1-dim'l. :(

$\lambda = 3$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = 0$
$v_2 = 0$
$v_3 = t$

$\vec{v} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

only 2 independent eigenvectors :(

it turns out that if λ_j is an eigenvalue of A, and if $(\lambda-\lambda_j)^{k_j}$ is the corresponding factor of $p(\lambda)$, then it's always true.

$$\dim(E_{\lambda_j}) \le k_j$$

$\uparrow$

λ_j eigenspace

<u>If</u> each $\dim(E_{\lambda_j}) = k_j$ (no defective eigenspaces), then
A is diagonalizable, and you get an eigenbasis by just combining bases for each eigenspace. (partial proof next page)

Are there times you know a matrix will be diagonalizable (even before you look for all the eigenvectors?)

<u>Theorem 1</u> Let $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ be eigenvectors (non-zero), of A, with different eigenvalues, i.e. $A\vec{v}_j = \lambda_j \vec{v}_j$ with $\lambda_j = \lambda_i$ only when $i=j$.

Then $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ are linearly independent.

<u>proof</u>: By induction on k: if $k=1$, we have one non-zero eigenvector $\vec{v}_1$, any non-zero vector is independent, since $c_1\vec{v}_1 = \vec{0} \Rightarrow c_1$ (some non-zero entry) $=0$ $\Rightarrow c_1 = 0$.

If the theorem is true for $k=m$ and we have $m+1$ vectors with distinct eigenvalues, consider

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m + c_{m+1}\vec{v}_{m+1} = \vec{0}$$

Compute $(A-\lambda_1 I)$ times this sum:

$$\Rightarrow \underbrace{c_1(\lambda_1-\lambda_1)}_{0}\vec{v}_1 + \underbrace{c_2(\lambda_2-\lambda_1)\vec{v}_2 + \dots + c_{m+1}(\lambda_{m+1}-\lambda_1)\vec{v}_{m+1}} = \vec{0}$$

here is a linear combo of ~~the~~ m vectors $\vec{v}_2, \dots \vec{v}_{m+1}$ which have distinct eigenvalues.

If the theorem is true for $k=m$ we deduce

$$c_2\underset{\neq 0}{(\lambda_2-\lambda_1)} = 0 = c_3\underset{\neq 0}{(\lambda_3-\lambda_1)} = \dots = c_{m+1}\underset{\neq 0}{(\lambda_{m+1}-\lambda_1)}$$

so $c_2 = c_3 = \dots = c_{m+1} = 0$

so $c_1\vec{v}_1 = 0$, so $c_1 = 0$

thus thm true for $k=1 \Rightarrow$ for $k=2 \Rightarrow$ for $k=3 \dots$ $\Rightarrow$ for every k

<u>Corollary</u> If $A_{n\times n}$ has n different eigenvalues, <u>then</u> A is diagonalizable.

<u>proof</u>. Let $\begin{matrix}\lambda_1, \lambda_2 \dots \lambda_n \\ \vec{v}_1, \vec{v}_2, \dots \vec{v}_n\end{matrix}$ be the "eigenpairs".

then $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$ are independent (by Thm 1), so a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$). ■

<u>Corollary</u> (see Theorem 4 page 382)

If the dimension of each eigenspace equals the power k that $(\lambda-\lambda_j)$ appears with in the characteristic polynomial, then A is also diagonalizable (as for $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$), and you get an eigenbasis by ~~tak~~ taking the union of your biases for each eigenspace