Math 2250-1
Tues 11/15

§ 6.1-6.2 eigenvalues and eigenvectors

this is a return to matrix algebra and linear algebra, which we will then use in Chapter 7 to study systems of differential equations

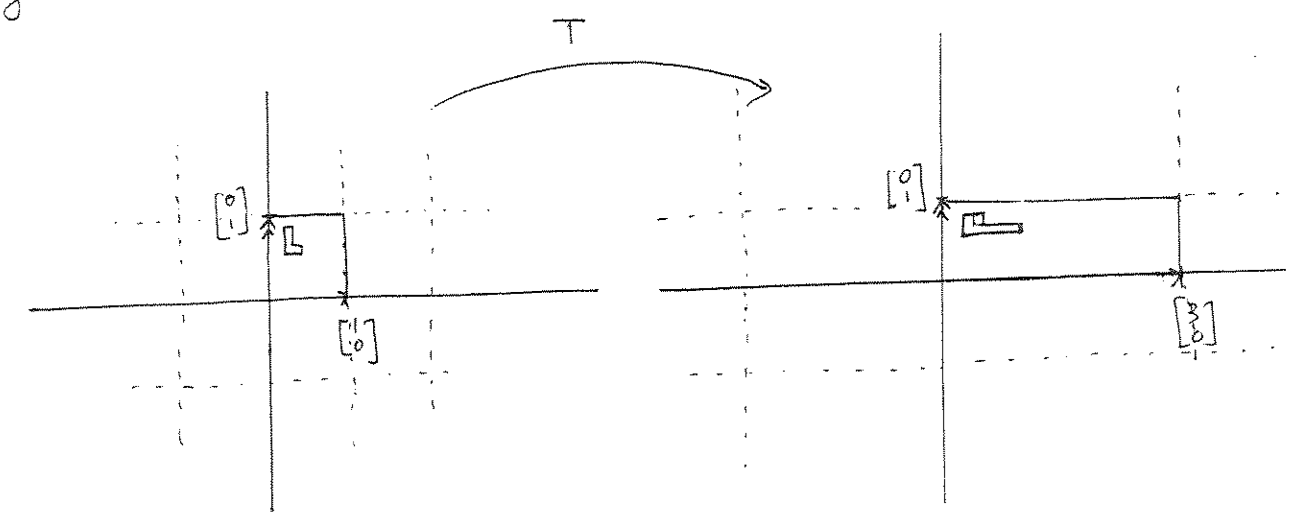
such as arise in coupled tank (input-output) models
coupled mass-spring systems, or
coupled circuit problems.

but for now we'll think geometrically.

Example Consider the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\vec{x}$

notice $A\vec{e}_1 = 3\vec{e}_1$
$A\vec{e}_2 = 1\vec{e}_2$

the fact that the transformation T multiplies the standard basis vectors by constants lets us understand the geometry of T:



T stretches by a factor of 3 in the $\vec{e}_1$ direction, and a factor of 1 in the $\vec{e}_2$ direction
linearity means the domain grid is converted into a similarly stretched range grid:

$$T(c_1\vec{e}_1 + c_2\vec{e}_2) = c_1 T(\vec{e}_1) + c_2 T(\vec{e}_2)$$
$$= c_1(3\vec{e}_1) + c_2\vec{e}_2 .$$

Exercise 1 Do a similar analysis for

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\vec{x}$$

Exercise 2 Same, for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Definition: If $A_{n\times n}$, and if $A\vec{v} = \lambda\vec{v}$ for scalar λ, $\vec{v} \neq \vec{0}$, then $\vec{v}$ is called an eigenvector of A, and λ is called the eigenvalue of $\vec{v}$.
(synonym: "characteristic vector")

In the three examples above, the eigenvectors were the standard basis vectors (or multiples of them), because the matrices were diagonal.
(And the eigenvalues were the corresponding diagonal entries.)
When A is not necessarily diagonal, eigenvectors of A can (still) be extremely useful. But how can you find them? ...

If $A\vec{v} = \lambda\vec{v}$ $\quad(\vec{v} \neq \vec{0})$,

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$A\vec{v} - \lambda I\,\vec{v} = \vec{0}$$

$$(A - \lambda I)\,\vec{v} = \vec{0}$$

this means $(A-\lambda I)^{-1}$ does not exist!

so $\det(A-\lambda I) = 0$. (and if $\det(A-\lambda I)=0$, the solution space to $(A-\lambda I)\,\vec{v} = \vec{0}$ will be at least 1-dim'l, because $\operatorname{rref}(A-\lambda I) \neq I$.)

So,

algorithm to find eigenvalues and eigenvectors for a matrix A:

<u>Step 1</u> Compute $p(\lambda) = \det(A-\lambda I)$ this is a polynomial of degree n in the variable λ. It is called the "characteristic polynomial" of A (?!)

<u>step 2</u> for each root λ_j of $p(\lambda)$, find the solution space to the homogeneous matrix equation $(A-\lambda_j I)\vec{x} = \vec{0}$.

Pick a basis for this subspace (called the λ_j eigenspace). (These will be eigenvectors of A with eigenvalues λ_j).

<u>Exercise 3</u> Find the eigenvalues and corresponding eigenspace bases for

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

Use your work to understand the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Exercise 4 Find the eigenvalues and eigenspace bases for

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

step 1 $p(\lambda)$ & its roots

step 2 eigenspace bases

ans: $p(\lambda) = -(\lambda-2)^2(\lambda-3)$

$E_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$

$E_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$