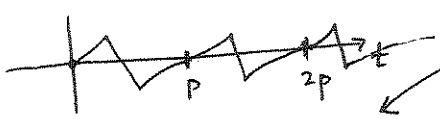
Math 2250-1
Mon. 11/14
§10.4–10.5, EP 7.6

Tuesday: §6.1–6.2 (also Wed.)

We continue to focus on the Laplace transform entries related to

	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$	
unit step function at $t=a$	$u(t-a)$	$\frac{e^{-as}}{s}$	§10.5
	$u(t-a)f(t-a)$	$e^{-as}F(s)$	§10.5
✓convolution	$f*g(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	§10.4 (this section is also where the entry $t\,f(t) \mid -F'(s)$ is located; we talked about this before)
	$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$	§10.5
δ "function" at $t=a$ (really δ operator)	$\delta(t-a)$	e^{-as}	EP 7.6

$f(t)$ periodic, period p:



$f(t)$ for $0 \le t \le p$; zero for $t \ge p$.

zero for $0 \le t \le p$; $f(t-p) = f(t)$ for $t \ge p$

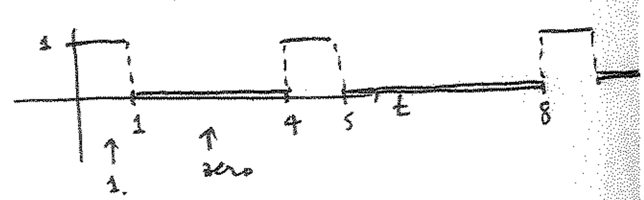
notice $f(t) = f(t)\left(1 - u(t-p)\right) + u(t-p)f(t-p)$

$$\Rightarrow \quad F(s) = \int_0^p e^{-st} f(t)\,dt + e^{-ps}F(s)$$

$$\Rightarrow \quad F(s)\left(1 - e^{-ps}\right) = \int_0^p e^{-st} f(t)\,dt$$

$$\boxed{F(s) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt}$$

Exercise 1: Find the Laplace transform the function whose graph is indicated, with period equal to 4:



ans $\dfrac{1-e^{-s}}{s(1-e^{-4s})}$

E.P. 7.6
impulses and the δ operator:

Consider a force $f(t)$ acting only on a very short time interval $a \le t \le a+\varepsilon$
e.g. a bat hitting a ball.

The impulse of the force is defined to be

$$p = \int_a^b f(t)\,dt$$

and this measures the net change in momentum of the object being forced, because

$$m v'(t) = f(t) \qquad \text{Newton 2}$$

$$\int_a^{a+\varepsilon} m v'(t)\,dt = \int_a^{a+\varepsilon} f(t)\,dt = p$$

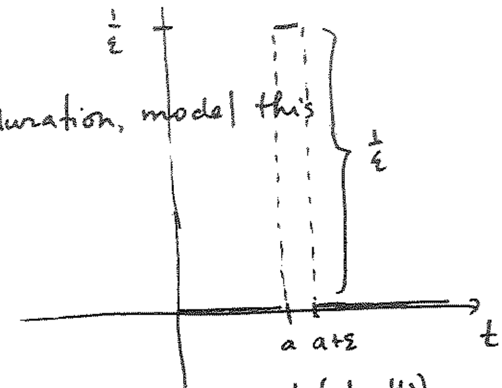
$$\| $$

$$\underbrace{m v(a+\varepsilon) - m v(a)}_{\text{change in momentum}}$$

Since p only depends on the integral of $f(t)$ over its duration, model this impulsive force as $p \cdot d_{a,\varepsilon}(t)$

unit ε-duration impulse

$$d_{a,\varepsilon}(t) = \begin{cases} \frac{1}{\varepsilon} & a \le t \le a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$



graph ($d_{a,\varepsilon}(t)$).

$$\int_a^{a+\varepsilon} d_{a,\varepsilon}(t)\,dt = \frac{1}{\varepsilon}\cdot\varepsilon = 1$$

for instantaneous force of impulse p
we'd like to let $\varepsilon \to 0$, of $p\,d_{a,\varepsilon}(t)$
This doesn't exactly work in the space of functions $f(t)$
but works great in Laplace transform space

(the limit would be 0 except at $t=a$ where it would be ∞, and its integral would be $p\cdot 1$)

$$d_{a,\varepsilon}(t) = \frac{1}{\varepsilon}\left(u(t-a) - u(t-(a+\varepsilon))\right)$$

$$\mathcal{L}\{d_{a,\varepsilon}(t)\}(s) = \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right) = e^{-as}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right)$$

$$\lim_{\varepsilon\to 0}: \quad e^{-as}\left(\lim_{\varepsilon\to 0} \frac{--\varepsilon}{\varepsilon}\right) \quad \text{L'Hopital}$$

$$= e^{-as}$$

$f(t)$	$F(s)$
$\delta(t)$	e^{-as}

unit instantaneous impulse → "δ function" example :

revisit the swingset from Friday

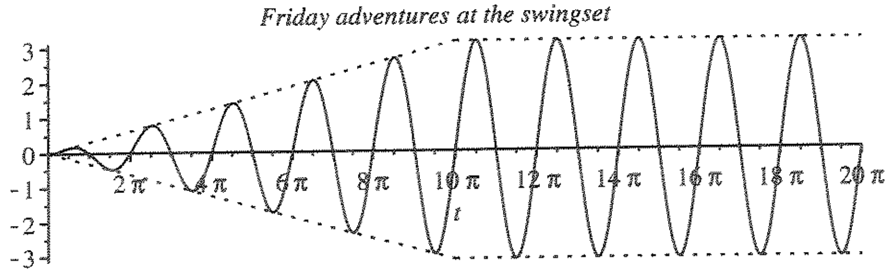
<u>Exercise 2</u> solve for $x(t)$

$$\begin{cases} x''(t) + x(t) = .2\pi\left[\delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + \delta(t-8\pi)\right] \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

Delta function and periodic function forcing of mechanical and electrical systems.
Sections 10.4, 10.5, EP 7.6

```
> with(plots) :
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(-Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(-.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = 'Friday adventures at the swingset');
```

Friday adventures at the swingset

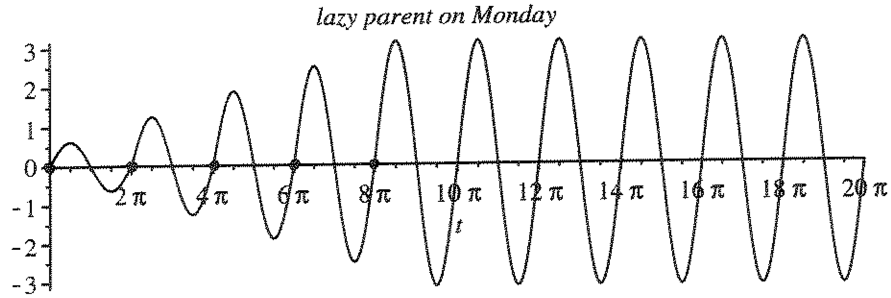



```
>
```

impulse solution: five equal impulses to get same final amplitude of π meters:

```
> f := t→.2·Pi·sum(Heaviside(t − k·2·Pi)·sin(t − k·2·Pi), k = 0 ..4) :
> plot(f(t), t = 0 ..20·Pi, color = black, title = 'lazy parent on Monday');
```

lazy parent on Monday

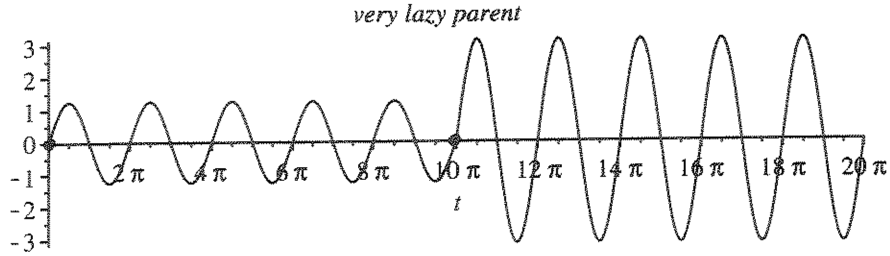



```
>
```

Or, an impulse at $t = 0$ and another one at $t = 10\pi$.

```
>
> g := t→.2·Pi·(2·sin(t) + 3·Heaviside(t − 10·Pi)·sin(t − 10·Pi)) :
> plot(g(t), t = 0 ..20·Pi, color = black, title = 'very lazy parent');
```

very lazy parent

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>
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