Math 2250-1
Friday 11/11

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Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$	e^{at}	$\dfrac{1}{s-a}$
$af(t) + bg(t)$	$aF(s) + bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s) - f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$ unit step function on/off forcing	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$ for $\mathcal{L}^{-1}\{F(s)G(s)\}(t)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$ "convolution integral"	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	unit step function $u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$ periodic forcing	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$	delta function for impulse forcing.	
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		

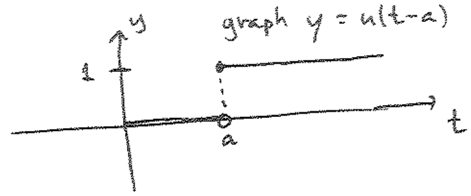
Today and Monday we will discuss the circled table entries, and their applications to forced oscillation problems

The unit step function:

$$u(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{and} \quad u(t-a) = \begin{cases} 1 & t-a \geq 0, \text{ i.e. } t \geq a \\ 0 & t-a < 0, \text{ i.e. } t < a \end{cases}$$

Maple calls this function "Heaviside"

graph $y = u(t-a)$



Exercise 1 Check $\mathcal{L}\{u(t-a)\}(s) = \dfrac{e^{-as}}{s}$

Exercise 2 Check the translation (in t-space) Laplace table entry

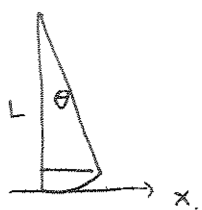
$$\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$$

Setup: an under-employed mathematician/engineer/scientist
(your choice)

likes to take his/her child to the swings ...

recall pendulum (linearized) eqtn, without forcing, for $\theta = \theta(t)$

$$L\theta'' + g\theta = 0$$



$\dashrightarrow x'' + g\frac{x}{L} = 0$

$\dashrightarrow mx'' + \frac{mg}{L}x = F_0 \cos\omega t \quad \leftarrow$ parent forcing (?!)

$x(t) = L\sin\theta(t)$
$\approx L\theta$ for small θ

so $x'' \approx L\theta''$

$\dashrightarrow x'' + \frac{g}{L}x = \frac{F_0}{m}\cos\omega t$

$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$

for resonance $\omega = \omega_0$
construct swing with $L = g \approx 9.8$ m.
so $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds

:)

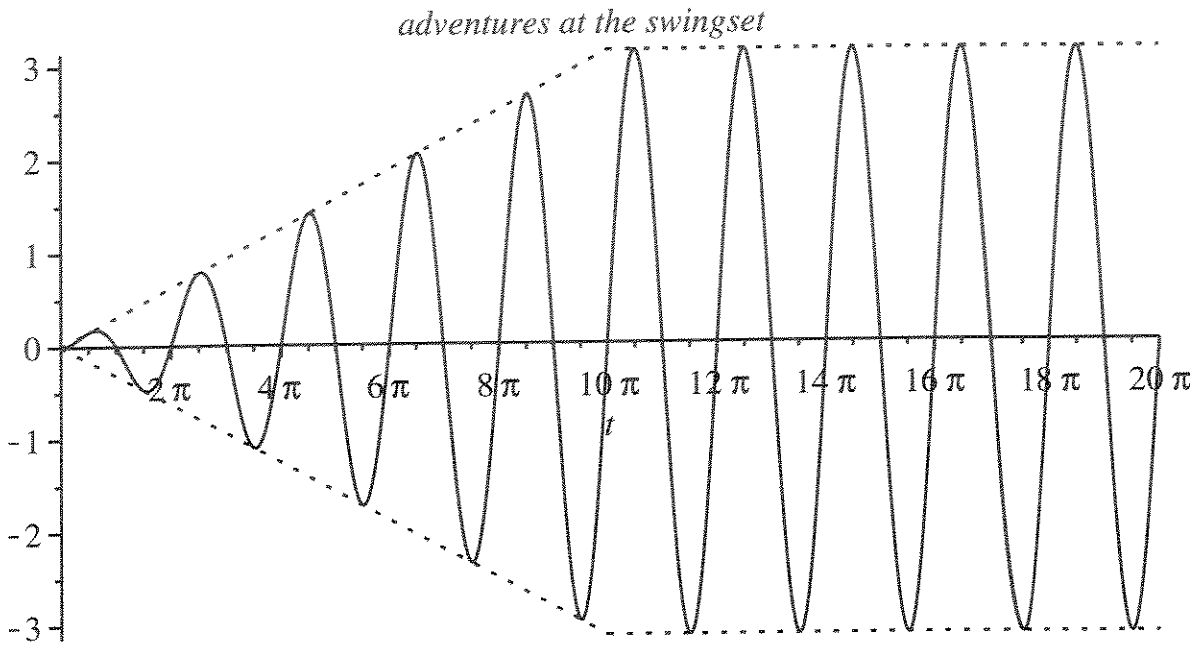
parent pushes sinusoidally for exactly 5 cycles, and with $\frac{F_0}{m} = .2$ and then releases:

$$\begin{cases} x''(t) + x(t) = .2\cos t\,(1 - u(t - 10\pi)) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

<u>Exercise 3</u>: Find $x(t)$. Show that after pushing has stopped, child is swinging with an amplitude of π meters.

Pictures for the swing.

```
> with(plots) :
> plot1 := plot(.1·t·sin(t), t=0..10·Pi, color=black) :
  plot2 := plot(Pi·sin(t), t=10·Pi..20·Pi, color=black) :
  plot3 := plot(Pi, t=10·Pi..20·Pi, color=black, linestyle=2) :
  plot4 := plot(-Pi, t=10·Pi..20·Pi, color=black, linestyle=2) :
  plot5 := plot(.1·t, t=0..10·Pi, color=black, linestyle=2) :
  plot6 := plot(-.1·t, t=0..10·Pi, color=black, linestyle=2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title='adventures at the swingset');
```



alternate approach via Chapter 5 (or Chapter 10).

step 1 solve $\begin{cases} x'' + x = .2\cos t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$ for $0 \le t \le 10\pi$

step 2 solve $\begin{cases} y'' + y = 0 \\ y(0) = x(10\pi) \\ y'(0) = x'(10\pi) \end{cases}$ for $\tilde{t} \geq 0$, and set $t = \tilde{t} + 10\pi$.

Laplace transform convolution

$$f*g\,(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$= g*f\,(t)\ !$$

(check by substituting $\tilde{\tau} = t-\tau$ in 1st integral)

Theorem

$$\mathcal{L}\{f*g\,(t)\}(s) = F(s)G(s)$$

useful for inverting products of Laplace transforms

example Verify the theorem for

$f(t) = \sin t$
$g(t) = \cos t$

(which will give another derivation of

$f(t)$	$F(s)$
?	$\frac{s}{(s^2+1)^2}$
$\cos t$	$\frac{s}{s^2+1}$
$\sin t$	$\frac{1}{s^2+1}$

)

- compute $f*g\,(t)$!

Hint, you may need to use $\sin^2\tau = \frac{1-\cos 2\tau}{2}$ trig identity

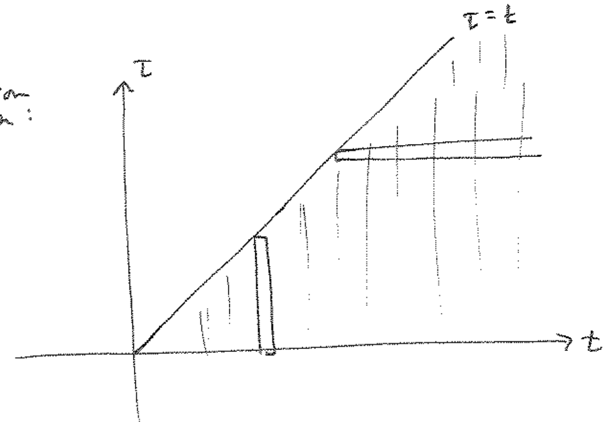
we'll do some fun convolution applications Monday, then start Chapter 6 proof of convolution theorem:

(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right) dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau)\,g(t-\tau)\,d\tau\,dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\,g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\, e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right] d\tau$$

$\tilde{t} = t - \tau$

$d\tilde{t} = dt$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}\right]}_{G(s)}$$

$$= G(s)\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)\,F(s) \qquad !!$$