Math 2250-1
Wed 12/7

9.3-9.4 cont'd

- finish the undamped rigid rod pendulum analysis, pages 2-5 Tuesday's notes.

Here are details we worked out yesterday:

$$\theta'' + \frac{g}{L}\sin\theta = 0$$

$$\downarrow \begin{array}{l} x = \theta(t) \\ y = \theta'(t) \end{array}$$

$$x' = y$$
$$y' = -\frac{g}{L}\sin x$$

equil. soltns $\begin{array}{l} y=0 \\ \sin x = 0 \end{array} \Rightarrow (n\pi, 0) \quad n \in \mathbb{Z}.$

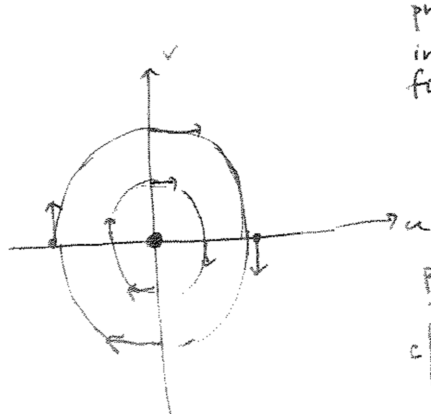
$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x & 0 \end{bmatrix}$$

if $x = n\pi$, n even, $\cos x = +1$

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$|J - \lambda I| = \lambda^2 + \frac{g}{L} = 0$$

$\left.\begin{array}{l} \lambda = \pm i\sqrt{\frac{g}{L}} \\ \quad = \pm i\omega_0 \end{array}\right\}$ stable center for linearized problem, indeterminant for non-linear



pt.	tang vector
$c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$c\begin{bmatrix} 0 \\ -g/L \end{bmatrix}$
$c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

if $x = n\pi$, n odd, $\cos x = -1$

$$J = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

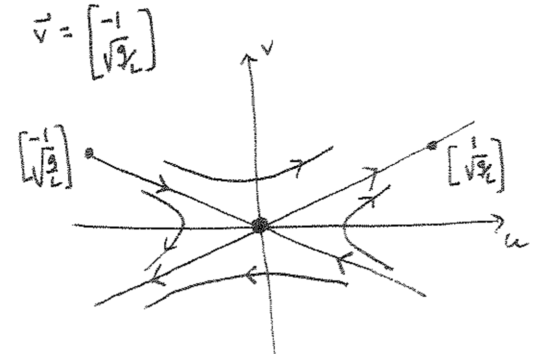
$$|J - \lambda I| = \lambda^2 - \frac{g}{L} = 0$$

$$\lambda = \pm\sqrt{\frac{g}{L}}$$

Saddle for linearized & non-linear system

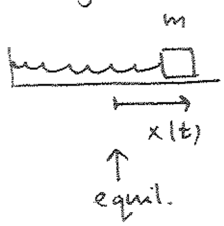
$\lambda = \sqrt{\frac{g}{L}} \qquad \lambda = -\sqrt{\frac{g}{L}}$

$\vec{v} = \begin{bmatrix} 1 \\ \sqrt{g/L} \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -1 \\ \sqrt{g/L} \end{bmatrix}$



- After recalling page 1, complete pages 3-5 Tuesday's notes

Then consider a non-linear modification of the mass-spring configuration



$$\boxed{m x''(t) = F(x)}$$

$$m x''(t) = -kx + \alpha x^2 + \beta x^3 + \dots$$

linear model : $m x''(t) = -kx$

assumes perfect Hooksian spring.

improvements: consider more terms in Taylor series, could model a variety of situations.

→ if force is an odd function, $F(-x) = -F(x)$, then only odd powers appear in Taylor series, and a good model might be

$$m x'' = -kx + \beta x^3$$

$\beta < 0$ "hard spring" (resists large deviations x)
$\beta > 0$ "soft spring"

→ force might not be odd, in which case

$m x'' = -kx + \alpha x^2$ might be a good model (resists $x < 0$ more than $x > 0$)

First order system for $m x''(t) = F(x)$

$$\boxed{\begin{aligned} x' &= y \\ y' &= \tfrac{1}{m} F(x). \end{aligned}}$$

so $x(t)$ = position
$y(t)$ = velocity.

Because the Force does not depend on velocity these systems always have a conserved total energy function. It can be recovered either using the concept of work from Calculus / Physics, or with the $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ separable DE's trick.

$$TE = KE + PE$$
$$= \tfrac{1}{2} m y^2 + \int_0^x -F(s)\,ds$$
$$\frac{d}{dt}(TE) = m y y' - F(x)\,x'(t)$$
$$= \cancel{m} x' \left(\tfrac{1}{\cancel{m}} F\right) - F x' \equiv 0$$

$$\frac{dy}{dx} = \frac{\frac{1}{m} F(x)}{y}$$
$$m y\, dy = F(x)\, dx$$
$$\tfrac{1}{2} m y^2 = \int F(x)\,dx + C$$

<u>Exercise 1</u> : Consider the "soft spring" problem

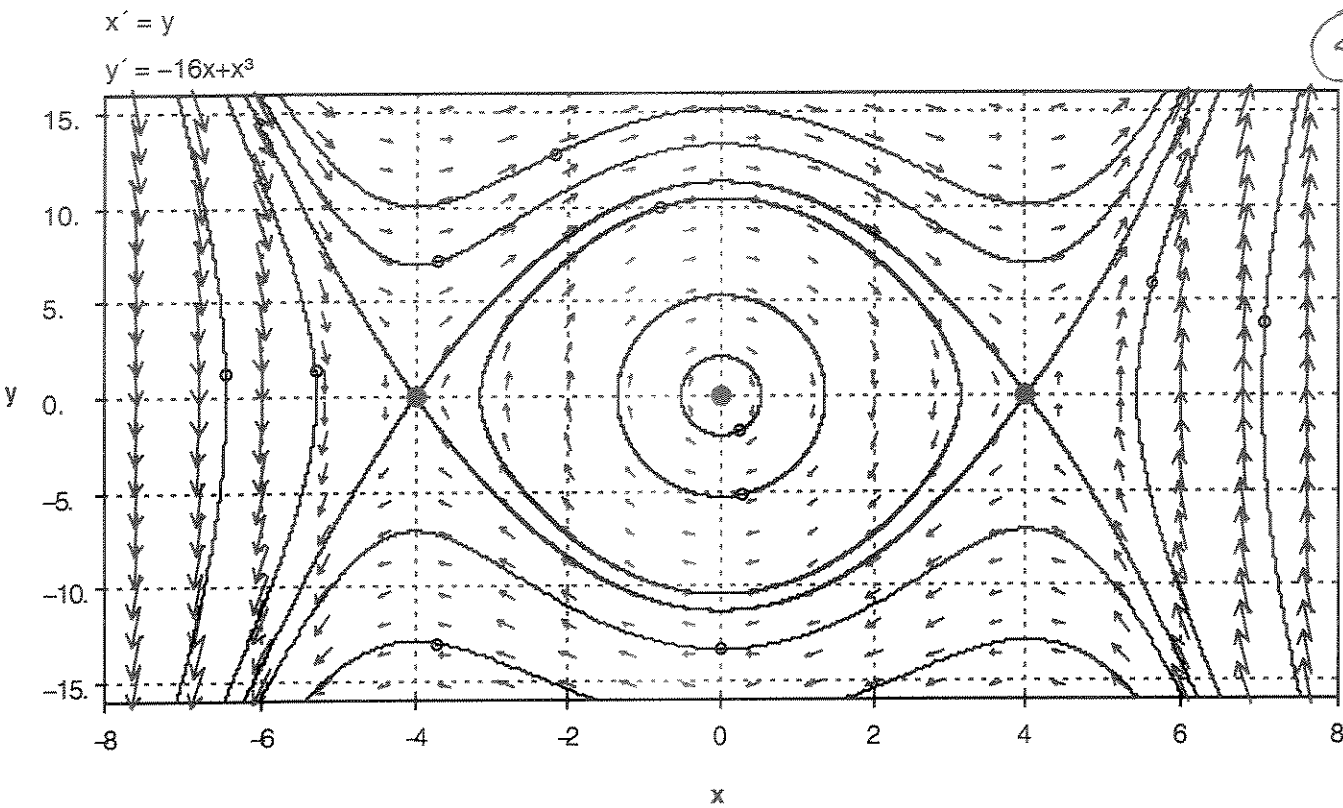
$$x''(t) = -16x + x^3$$

a) Convert this to an equivalent 1st order system

b) Find the equilibrium solutions

c) Verify that the linearizations are consistent with the phase portrait, at the equilibria.

d) What is the total (conserved) energy in this problem?



soft spring total energy function

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> with(plots) :
> plot3d(y^2/2 + 8·x^2 − x^4/4, x=-6..6, y=-16..16, axes = boxed);
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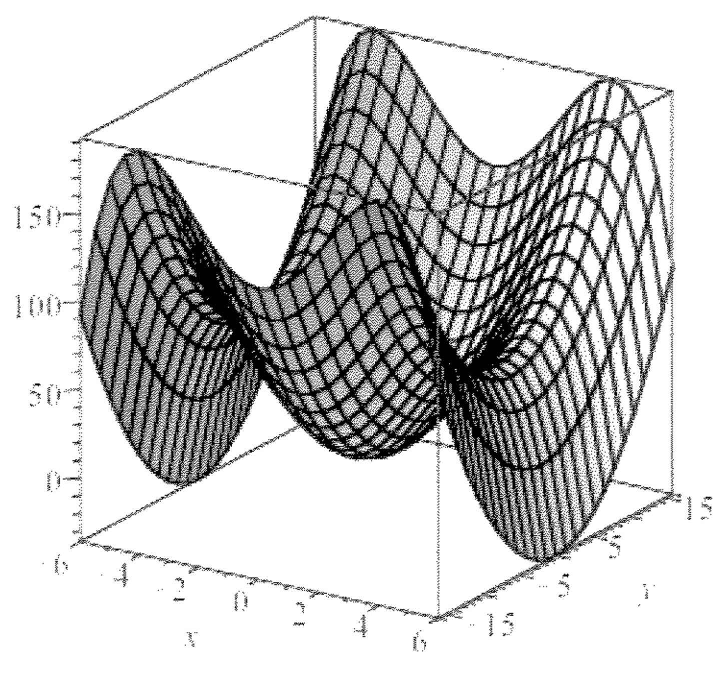
$$plot3d\left(\frac{y^2}{2} + 8\cdot x^2 - \frac{x^4}{4}, x=-6..6, y=-16..16, axes = boxed\right);$$



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