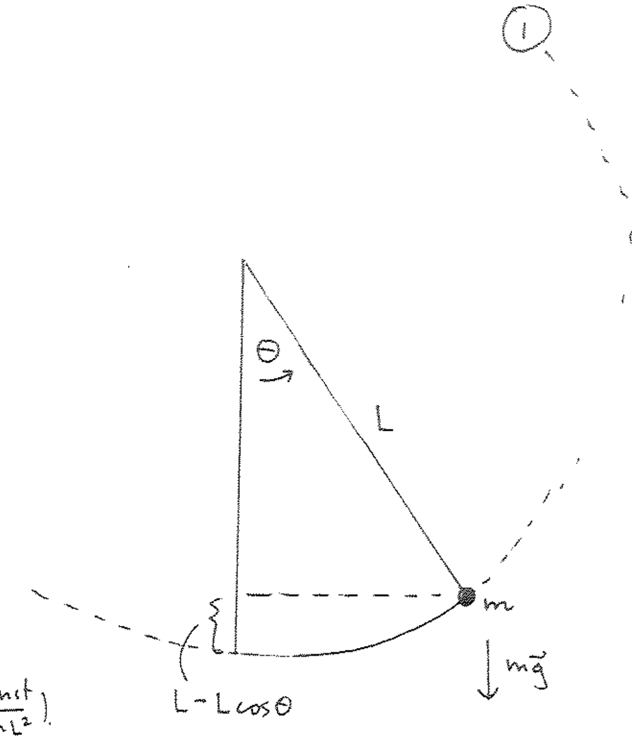
Math 2250
Tues 12/6 9.3–9.4

- Finish Exercise 2 Monday: predator-prey problem

Then consider a mechanical example from §9.4: undamped rigid-rod pendulum



(1) $\theta''(t) + \frac{g}{L}\sin\theta = 0$

Recall our derivation:

$$KE + PE = \text{constant}$$

$$\frac{1}{2}m(L\theta')^2 + mg\underbrace{L(1-\cos\theta)}_{\text{height from bottom}} = \text{const.}$$

(2) $\div mL^2$: $\frac{1}{2}(\theta')^2 + \frac{g}{L}(1-\cos\theta) = C \quad \left(= \frac{\text{const}}{mL^2}\right)$

We took $\frac{d}{dt}$ of (2) to get (1).

So, for $x(t) = \theta(t)$
$y(t) = \theta'(t)$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g}{L}\sin x \end{bmatrix}$$

equil. sol'tns: $y = 0$, $\sin x = 0$ $\Rightarrow (0,0), (\pm\pi, 0), (\pm 2\pi, 0) \ldots$

<u>Exercise 1</u> Describe what the equilibrium sol'tns have to do with pendulum configurations

$x' = y$
$y' = -\frac{g}{L}\sin x$

equil soltns : $(x,y) = (n\pi, 0)$ $\quad n \in \mathbb{Z}$.

Exercise 2 : Linearize : Find the Jacobian matrix. Classify the equilibrium soltns (if you can)

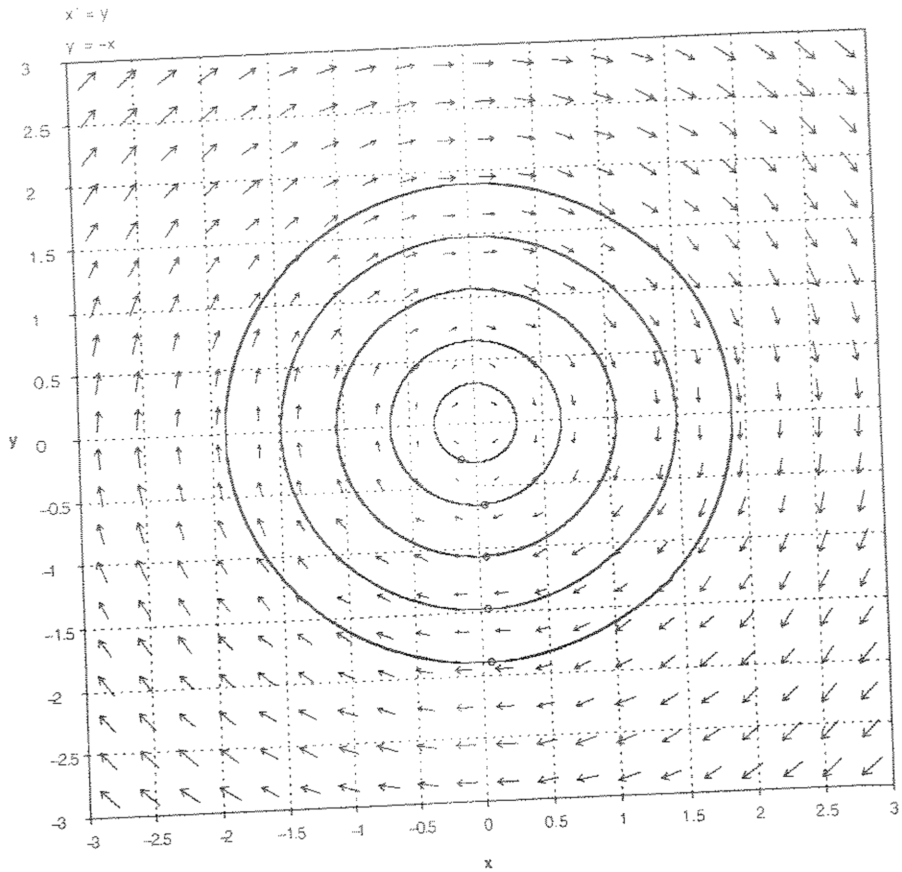
Separate out the cases

$x = n\pi$ n even $\qquad\qquad$ $x = n\pi$ n odd

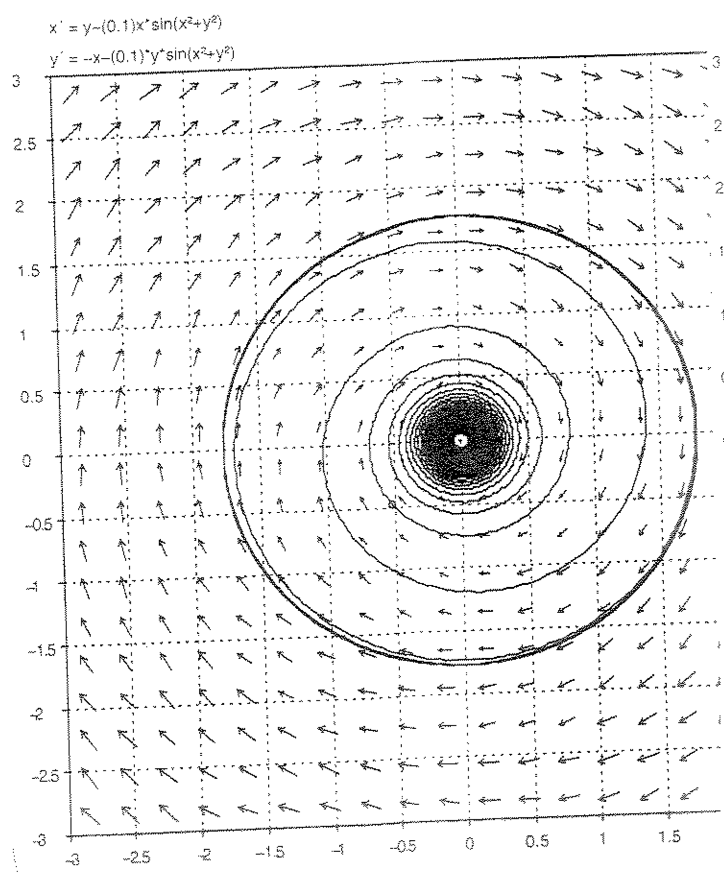
Exercise 3 : Let $\frac{g}{L} = 1$ (for simplicity and because it fits the parameters of our swingset from before).

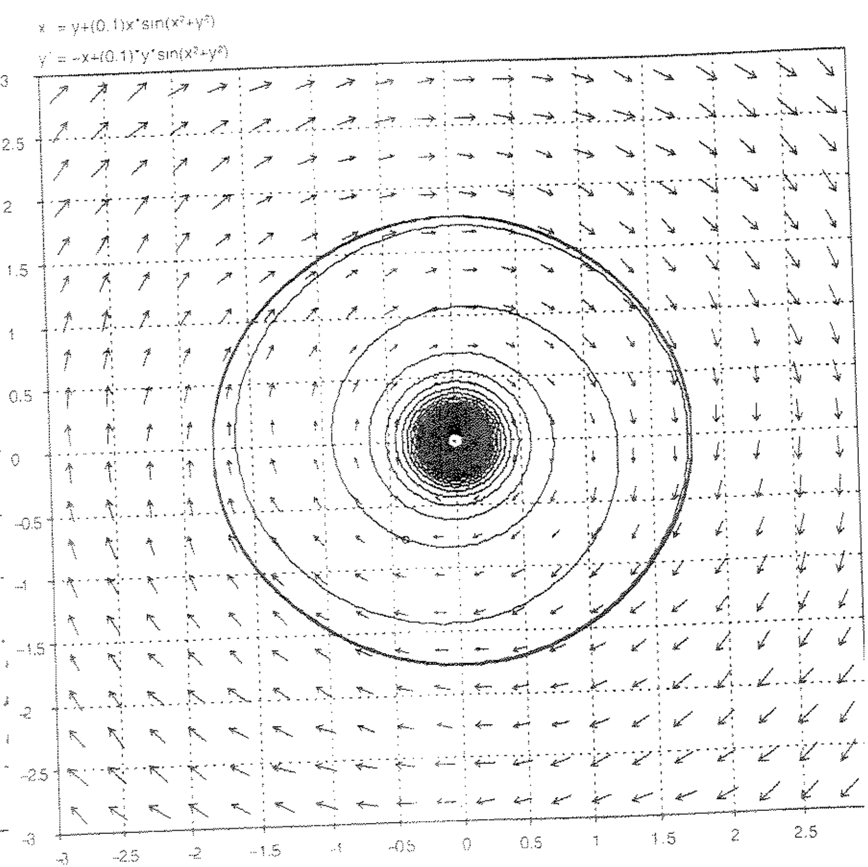
Show that for the linearized problem at $(0,0)$ (or $(n\pi, 0)$, n even), (stable center)
the solutions are given by

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} C\cos(t-\alpha) \\ C\sin(t-\alpha) \end{bmatrix}$$



stable center for linearization
does not imply stability/instability for
non-linear system





In these borderline cases you might be saved by physics or separable DE's, in trying to determine stability.

$$\frac{dx}{dt} = y \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{g}{L}\sin x}{y}$$

$$\frac{dy}{dt} = -\frac{g}{L}\sin x$$

$$\frac{dy}{dx} = \frac{-g\sin x}{Ly}$$

$$y\,dy = -\frac{g}{L}\sin x\,dx$$

$$\frac{1}{2}y^2 = \frac{g}{L}\cos x + C$$

$$\frac{1}{2}y^2 - \frac{g}{L}\cos x = C$$

$\longleftarrow$ this is the conservation of energy equation (2) on page 1, with a different "C"!

Notice that $E(x,y) := \frac{1}{2}y^2 - \frac{g}{L}\cos x$

has global minimum values at $y=0$, $x=0, \pm 2\pi, \pm 4\pi, \ldots$

$E(x,y) \geq -\frac{g}{L}$, and $E(x,y) = -\frac{g}{L}$ only at $(0, n\pi)$ n even.

Thus, solutions that start close to these equilibria must stay close, by conservation of energy!

In fact, all of these nearby solutions are periodic, as you can see from the graph of the function $E(x,y)$ and its contours, at constant height.

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> with(plots):
> plot3d(1/2*y^2 - cos(x), x = -8..8, y = -2..2, color = black, style = line, axes = boxed, title
    = `graph of the total energy function for undamped rigid rod pendulum`);
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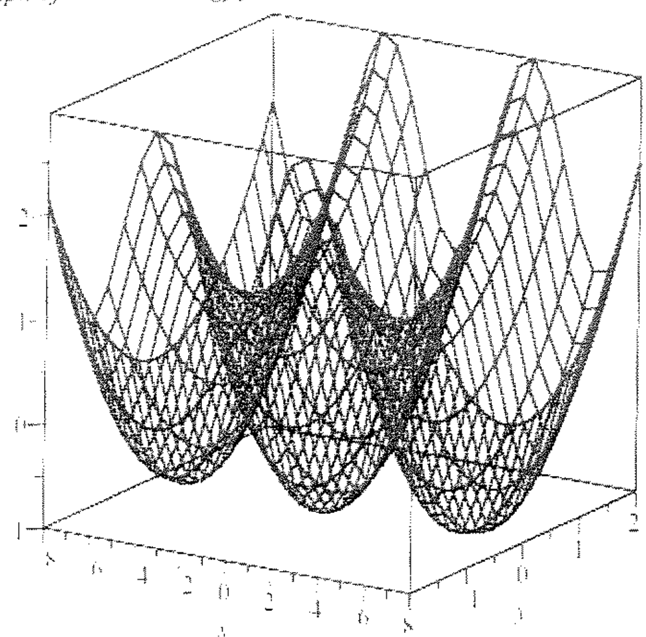
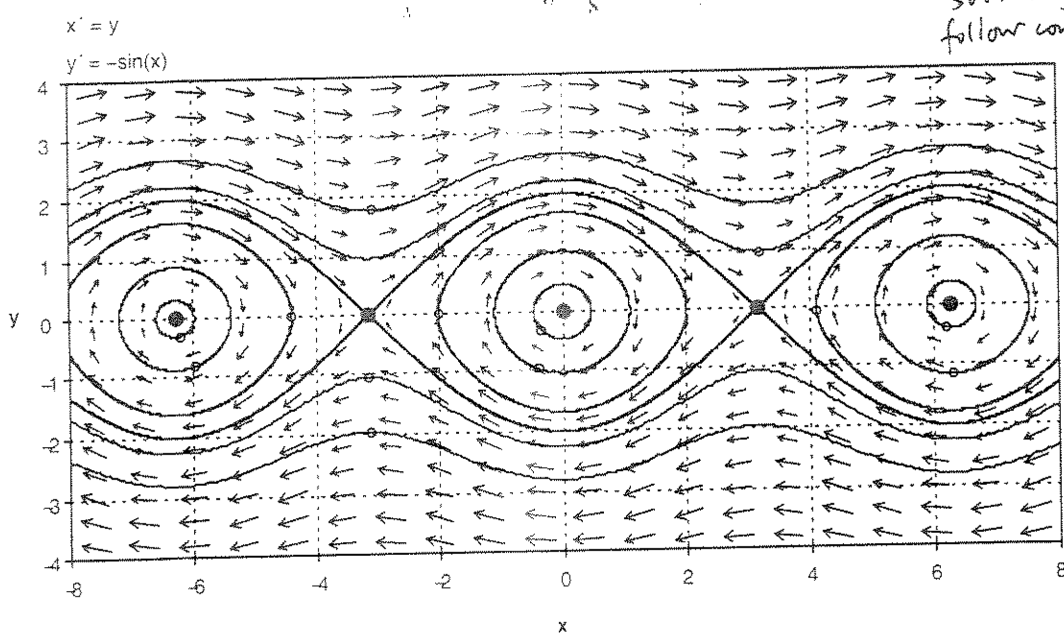
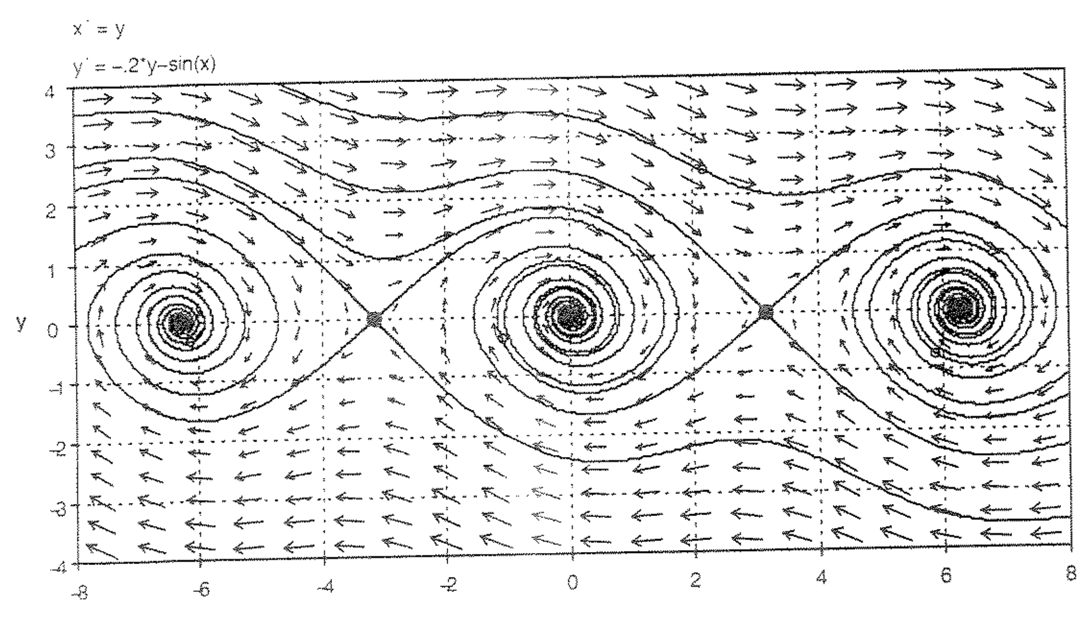
graph of the total energy function for undamped rigid rod pendulum



sol'n trajectories follow contours of total energy function





damping! see HW.